Ramanujan Reaches His Hand From His Grave To Snatch Your Theorems From You

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Abstract. Because many of Ramanujan's theorems were hidden from the public for many years, it was natural that other mathematicians would unknowingly rediscover some of his unpublished work. We give examples of theorems from the theory of the Riemann zeta function, summation formulas, prime number theory, combinatorics, and partitions that have names of others attached to them, but they were discovered much earlier by Ramanujan.

1. Introduction

Born on December 22, 1887, India's greatest mathematician, Srinivasa Ramanujan, began to record his discoveries in notebooks in about 1904 when he entered the Government College of Kumbakonam for what was to be only one year of study. For the next five years, Ramanujan did mathematics, mostly in isolation, while logging his findings without proofs in notebooks. Ramanujan moved to Madras in 1910 and, while working as a clerk in the Madras Port Trust Office, was encouraged by the Manager and Chief Accountant, S. Narayana Aiyar, and the Chairman, Sir Francis Spring, to write to English mathematicians about his work. After communicating about 120 of his theorems to G. H. Hardy in early 1913, Ramanujan accepted Hardy's invitation to go to Cambridge, and so on March 17, 1914, Ramanujan departed India for England. At about that time, from letters that he wrote to friends back home in Madras [15, pp. 112–113; 123–125], Ramanujan ceased recording his theorems in notebooks to concentrate on publishing research that he was conducting in England. After returning to India in March, 1919, Ramanujan began to log entries in what was later to be called *Ramanujan's Lost Notebook*, which was found by George Andrews in the library at Trinity College, Cambridge in March, 1976. After a lengthy illness, which had confined him to nursing homes during his last two years in England, Ramanujan died on April 26, 1920 at the age of 32.

Ramanujan's earlier notebooks were not published until 1957 [30], and his later lost notebook was not published until 1988 [31]. Thus, not surprisingly, it transpired that some of Ramanujan's discoveries were hidden for many years and were in the meantime proved by others.

On June 1–5, 1987, a meeting was held at the University of Illinois to commemorate the centenary of Ramanujan's birth and to survey the many areas of mathematics (and of physics) that have been profoundly influenced by his work. One of the speakers, R. William Gosper, remarked in his lecture, "How can you love this man? He continually reaches his hand from his grave to snatch your theorems from you." The purpose of this paper is to provide just a few of the many examples for which others have received credit for theorems, but unknown to them, their discoveries were not original with them; they were first unearthed by Ramanujan.

2. Ramanujan's Early Work

A perusal of issues of the *Journal of the Indian Mathematical Society* from its inception in 1907 to, say, the late 1920's shows that some of the earlier

Paying homage to Ramanujan: Professor Berndt holding the slate used by Ramanujan for proving his numerous discoveries.
(Photo taken in October, 2002 at the home of S. Narayanan, a grandson of S. Narayana Aiyar.)

papers and problems that Ramanujan submitted to the *Journal of the Indian Mathematical Society* were much in the spirit of mathematics popular in India at that time. Thus, it is not unexpected that some articles published in the *Journal* in the decade after Ramanujan's death might contain entries found many years later to be in Ramanujan's notebooks. In particular, two papers by M. B. Rao and M. V. Ayyar [32] and one by S. L. Malurkar [26] contain several entries from Chapter 14 in Ramanujan's second notebook [30], [9]. We offer one example from a paper by Malurkar, who studied physics at Cambridge University and who was Director of the observatories in Pune and Mumbai for much of his career.

Theorem 2.1. *Let* $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, Re $s > 1$, *denote the Riemann zeta function, and let* B_n, $n \geq 0$, *denote the nth Bernoulli number. If* α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$, *and if* r *is a positive integer, then*

$$(4\alpha)^{-r}\left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\alpha}-1)}\right) - (-4\beta)^{-r}\left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\beta}-1)}\right) = -\sum_{k=0}^{r+1} \frac{(-1)^k B_{2k} B_{2r+2-2k} \alpha^{r+1-k} \beta^k}{(2k)!(2r+2-2k)!}. \tag{2.1}$$

Theorem 2.1 can be found as Entry 21(i) in Chapter 14 in Ramanujan's second notebook [9, pp. 275–276]. It also appears in a manuscript of Ramanujan that was published for the first time in its original handwritten form with his lost notebook [31, pp. 319–320, formula (28)]. This manuscript, along with commentaries, can also be found in [12] and [1]. The special case $\alpha = \beta = \pi$ was first proved by M. Lerch in 1901 [25]. There are now several proofs of Theorem 2.1, and references to these many proofs can be found in [9, p. 276] and [1]. An extensive generalisation of Entry 2.1 can be found in Entry 20 of Chapter 16 in Ramanujan's first notebook [30], [10, pp. 429–432], and another one in [11]. Moreover, there are further generalisations, including analogues for Dirichlet L-functions.

Ramanujan's formula (2.1) is fascinating for several reasons. It is well known that, for each positive integer n, $\zeta(2n)$ can be explicitly evaluated as a rational multiple of π^{2n} containing the factor B_{2n}. However, except for the fact that $\zeta(3)$ is irrational, which was first proved by R. Apéry [3], we currently know no further information about the arithmetical nature of $\zeta(2n+1)$. If we set $r = 2n+1$, $n \geq 0$, and $\alpha = \beta = \pi$ in (2.1), we find that $\zeta(4n+3)$ can be represented as a rational multiple of π^{4n+3} involving Bernoulli numbers *plus* a rapidly convergent series. Thus, $\zeta(4n+3)$ is "almost" a rational multiple of π^{4n+3}.

3. Bell Numbers and Polynomials

Most readers will be familiar with the name of E. T. Bell because of his popular book [7]. On the other hand, those with a combinatorial interest will identify him with Bell numbers and Bell polynomials. However, these numbers and polynomials should have Ramanujan's name attached to them, and not Bell's, because Ramanujan extensively studied these numbers and polynomials in Chapter 3 of his second notebook [30], [8], probably over 30 years before Bell wrote his papers [5], [6] on these numbers and polynomials.

The Bell polynomials $\varphi_n(x)$, $n \geq 0$, are defined by

$$\varphi_0(x) \equiv 1, \qquad e^x \varphi_{n+1}(x) := \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}, \qquad n \geq 0.$$

Readers should verify that $\varphi_n(x)$ is a polynomial of degree n, and, in particular,

$$\begin{aligned}
\varphi_1(x) &= x,\\
\varphi_2(x) &= x + x^2,\\
\varphi_3(x) &= x + 3x^2 + x^3,\\
\varphi_4(x) &= x + 7x^2 + 6x^3 + x^4.
\end{aligned}$$

They can be generated by the exponential generating function

$$e^{x(e^a-1)} = \sum_{n=0}^{\infty} \frac{a^n}{n!}\varphi_n(x).$$

The nth Bell number $B(n)$, $n \geq 1$, is defined by

$$B(n) = \varphi_n(1).$$

For example,

$$B(1) = 1, \quad B(2) = 2, \quad B(3) = 5,$$

$$B(4) = 15, \quad B(5) = 52, \quad B(6) = 203.$$

Combinatorially, $B(n)$ is equal to the number of ways of partitioning a set of n objects into subsets. In early editions of the Japanese classic, *The Tale of Genji*, written by Lady Shikibu Murasaki early in the 11th century, an arrangement of five incense sticks can be found at the beginning of each of the 54 chapters, except for the first and last. Thus, the author knew that the 5th Bell number equals 52. Readers enjoying poetry might like to know that the number of ways of rhyming a sonnet is $B(14) = 190{,}899{,}322$.

4. Koshliakov's Formula and Guinand's Formula

The Russian mathematician, N. S. Koshliakov (1891–1958), is chiefly remembered by a formula that now bears his name [24]. However, most of his work has been unfortunately neglected, and consequently his contributions to mathematics under appreciated. In 1942, he was arrested on fabricated charges and sent to a labour camp in the Ural mountains. Classified as an invalid after suffering from complete exhaustion, he found time to do mathematics, often under extremely difficult conditions, before being released in 1949.

Theorem 4.1 (Koshliakov's Formula)**.** *Let $K_0(z)$ denote the modified Bessel function of order 0, let $d(n)$ denote the number of positive divisors of the positive integer n, and let γ denote Euler's constant. If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(\frac{\gamma - \log(4\pi\alpha)}{\alpha} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\alpha)\right) = \sqrt{\beta}\left(\frac{\gamma - \log(4\pi\beta)}{\beta} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\beta)\right). \quad (4.1)$$

Why is Koshliakov's formula interesting? Recall that the theta transformation formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau} = \sqrt{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau}, \qquad \operatorname{Re}\tau > 0, \quad (4.2)$$

is the most common route to the functional equation of the Riemann zeta function

$$\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1}{2}(1-s)\right)\zeta(1-s). \quad (4.3)$$

In fact, (4.2) and (4.3) are equivalent. Koshliakov's formula (4.1) is an analogue of (4.2), and, as shown by W. L. Ferrar [20] and by F. Oberhettinger and K. L. Soni [28], is equivalent to the functional equation for $\zeta^2(s) = \sum_{n=1}^{\infty} d(n)n^{-s}$, which, of course, is obtained by squaring both sides of the functional equation (4.3).

Koshliakov published his proof of Theorem 4.1 in 1929 [24]. However, his formula can be found on page 253 of the publication containing the lost notebook [31], which was written in the last year of Ramanujan's life 1919–1920. The volume [31] contains further partial manuscripts and fragments written by Ramanujan; page 253 is such a page, and so it is possible that Ramanujan proved the formula earlier than 1919. Ramanujan clearly derived Koshliakov's formula from "Guinand's formula", which can be found on the same page, and which was first proved in print by A. P. Guinand [23] in 1955. Thus, on a single page, Ramanujan reached up his hand to snatch formulas that had brought fame to two mathematicians.

Theorem 4.2 (Guinand's Formula)**.** *Let $\sigma_k(n) = \sum_{d|n} d^k$, let $\zeta(s)$ denote the Riemann zeta function, and let $K_\nu(z)$ denote the modified Bessel function of order ν. If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if s is any complex number, then*

$$\begin{aligned}\sqrt{\alpha}\sum_{n=1}^{\infty}&\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)\\ &= \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2} - \alpha^{(1-s)/2}\}\\ &\quad + \frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2} - \alpha^{(1+s)/2}\}. \quad (4.4)\end{aligned}$$

Since $K_s(z) = K_{-s}(z)$, we see that (4.4) is invariant under the transformation $s \to -s$.

5. Snatching from Guinand Again

Ramanujan's lost notebook contains at least three further results that were first proved in print by Guinand. In this section we examine another beautiful transformation formula, thought first to have been established by Guinand [21], but found ensconced in a two-page partial manuscript published with the lost notebook [31, p. 220]. For further examples and details, see the forthcoming volume [1] by Andrews and the author.

To state this aforementioned formula, we require some definitions. Set

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots, \quad (5.1)$$

as $z \to \infty$, $|\arg z| < \pi$. Next, Riemann's ξ-function is defined by

$$\xi(s) := (s-1)\pi^{-s/2}\Gamma(1+\tfrac{1}{2}s)\zeta(s)$$

while the Riemann Ξ-function is given by

$$\Xi(t) := \xi(\tfrac{1}{2} + it).$$

Theorem 5.1. *Define*

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x.$$

If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\}$$
$$= \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \tag{5.2}$$

Note that the asymptotic formula (5.1) ensures the convergence of the infinite series in (5.2). The appearance of the Riemann zeta function on the far right side of (5.2) is unexpected. Ramanujan's recording of (5.2) is given as formula (7) on page 220 of [31] amidst several examples of Fourier sine and cosine transforms. In particular, formula (3) on that page is given as

$$\int_0^{\infty}(\psi(1+x) - \log x)\cos(2\pi nx)\,dx$$
$$= \frac{1}{2}(\psi(1+n) - \log n), \tag{5.3}$$

i.e., $\psi(1+x) - \log x$ is a self-reciprocal Fourier cosine transform. Guinand evidently discovered the first equality in (5.2) in the course of reading page proofs for [21], for he gives the identity in a footnote at the end of his paper. Furthermore, he writes that it "can be deduced from" (5.3). It is clear that both Ramanujan and Guinand employed (5.3) in their proofs. Referring to (5.2), Guinand asserts that "This formula also seems to have been overlooked." In [22], Guinand provides another proof of the first part of (5.2) that also depends on (5.3), about which he writes, "Professor T. A. Brown tells me that he proved the self-reciprocal property of $\psi(1+x) - \log x$ some years ago, and that he communicated the result to the late Professor G. H. Hardy. Professor Hardy said that the result was also given in a progress report to the University of Madras by S. Ramanujan, but was not published elsewhere." In March, 1913, Ramanujan received a scholarship from the University of Madras, with the only requirement being that he had to write quarterly reports about his research. Three such reports were written before he departed for England in March, 1914. Hardy's memory was perhaps flawed, because (5.3) cannot be found in Ramanujan's quarterly reports, which are detailed and discussed in [8], although it can be found in a partial manuscript [31, pp. 219–220], which evidently was also in Hardy's possession. The second equality in (5.2) was first proved by the author and A. Dixit [14], who gave two proofs of the first equality. Moreover, Dixit [17], [18] has established generalisations and analogues of (5.2).

6. Dickman's Function

Dickman's function $\rho(u)$, first introduced by K. Dickman in 1930 [16], plays a central role in prime number theory. For $0 \le u \le 1$, let $\rho(u) \equiv 1$. For each integer $k \ge 1$, $\rho(u)$ is defined inductively for $k \le u \le k+1$ by

$$\rho(u) = \rho(k) - \int_k^u \rho(v-1)\frac{dv}{v}.$$

Dickman's function is continuous at $u = 1$ and differentiable for $u > 1$. Equivalently, $\rho(u)$ can be defined by the differential-difference equation

$$u\rho'(u) + \rho(u-1) = 0, \qquad u > 1.$$

Let $P^{+}(n)$ denote the largest prime factor of the positive integer n, and set

$$\Psi(x,y) := |\{n \le x : P^{+}(n) \le y\}|. \tag{6.1}$$

On page 337 in his lost notebook [31], Ramanujan offers several asymptotic formulas for $\Psi(x, x^{\epsilon})$, although in a different language. We quote just one instance. "Let $\phi(x)$ denote the number of numbers of the form

$$2^{a_2}3^{a_3}5^{a_5}\cdots p^{a_p}, \qquad p \le x^{\epsilon},$$

not exceeding x. Then, for $\frac{1}{2} \le \epsilon \le 1$,

$$\phi(x) \sim x\left\{1 - \int_{\epsilon}^{1}\frac{du_0}{u_0}\right\}." \tag{6.2}$$

His next formula is for $\frac{1}{3} \le \epsilon \le \frac{1}{2}$, and so on. In the notation (6.1), Ramanujan's function $\phi(x) = \Psi(x, x^{\epsilon})$. Ramanujan hence proved Dickman's [16] famous asymptotic formula

$$\Psi(x, x^{1/u}) \sim x\rho(u), \qquad x \to \infty,$$

while, in fact, giving a representation for $\rho(u)$ in terms of integrals, as in the first instance (6.2). According to the author's colleague, A. J. Hildebrand, Ramanujan's theorem is equivalent to a folklore theorem, which we cannot find located in the literature.

Theorem 6.1. *Define, for $u \geq 0$,*

$$I_0(u) := 1, \qquad I_k(u) := \int_{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k \leq u}} \!\!\cdots\! \int \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k}, \qquad k \geq 1.$$

Then, for $u \geq 0$,

$$\rho(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} I_k(u). \tag{6.3}$$

The series on the right-hand side of (6.3) is finite, since if $k > u$, then $I_k(u) = 0$, for the conditions $t_1, \ldots, t_k \geq 1$ and $t_1 + \cdots + t_k \leq u$ are vacuous in this case. If we make the changes of variable $\epsilon = 1/u$ and $u_j = \epsilon t_j = t_j/u$, $1 \leq j \leq k$, then we obtain Ramanujan's theorem, the first instance of which is given in (6.2).

Excellent sources for information on the Dickman function and its prominence in prime number theory are G. Tenenbaum's treatise [33, Chapter III.5] and P. Moree's dissertation [27].

7. Ranks and Cranks

Let $p(n)$ denote the number of unrestricted partitions of the positive integer n. In 1944, F. Dyson [19] sought to find combinatorial explanations for Ramanujan's famous congruences [29]

$$p(5n+4) \equiv 0 \pmod 5,$$

$$p(7n+5) \equiv 0 \pmod 7,$$

$$p(11n+6) \equiv 0 \pmod{11}.$$

Accordingly, he defined the *rank of a partition* to be *the largest part minus the number of parts*. For example, the rank of $3+3+2+1$ is $3-4=-1$. Dyson observed that the congruences for the rank modulo 5 and 7 divided the partitions of $5n+4$ and $7n+5$, respectively, into equinumerous classes. These conjectures were subsequently proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [4]. However, for the third congruence, the corresponding criterion failed, and so Dyson conjectured the existence of a statistic, which he called the *crank* to combinatorially explain the congruence $p(11n+6) \equiv 0 \pmod{11}$. The *crank of a partition* was found by Andrews and Garvan [2] and is defined to be *the largest part if the partition contains no one's, and otherwise to be the number of parts larger than the number of one's minus the number of one's*. The crank divides the partitions into equinumerous congruence classes modulo 5, 7, and 11 for the three congruences, respectively.

In fact, in his lost notebook [31], Ramanujan had recorded the generating functions for both the rank and the crank. First, if $N(m,n)$ denotes the number of partitions of n with rank m, then

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n}. \tag{7.1}$$

Second, if $M(m,n)$ denotes the number of partitions of n with crank m, except for a few small values of m and n, then

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{7.2}$$

In (7.1) and (7.2),

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots, \qquad |q| < 1.$$

We do not know if Ramanujan knew the combinatorial implications of the rank and crank, but from the many results on these generating functions found in his lost notebook, it is clear that he had realised the importance of these two functions. Moreover, on page 184 in [31] he actually records an observation that is equivalent to Dyson's assertion about the congruences for the ranks of the partitions of $5n+4$.

Except for discerning an alternative formulation of a later discovery of Dyson, in this section we have strayed slightly from the theme of our paper to emphasise Ramanujan's *anticipation* of later, fundamental developments in the theory of partitions. It is also fitting to end our paper on this topic, because there is evidence that Ramanujan's very last mathematical thoughts were on cranks before he died on April 26, 1920 [13].

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