Mathematical Billiards

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This Letter presents some historical notes and some very elementary notions of the mathematical theory of billiards. We give the most interesting and popular applications of the theory.

1. Introduction

A billiard is a dynamical system in which a particle alternates between motion in a straight line and specular reflections from a boundary, i.e. the angle of incidence equals the angle of reflection. When the particle hits the boundary it reflects from it without loss of speed. Billiard dynamical systems are Hamiltonian idealisations of the game of billiards, but where the region contained by the boundary can have shapes other than rectangular and even be multidimensional. In Fig. 1 some examples of mathematical billiard tables and trajectories are shown.

Dynamical billiards may also be studied on non-Euclidean geometries; indeed, the very first studies of billiards established their ergodic motion on surfaces of constant negative curvature. The study of billiards which are kept out of a region, rather than being kept in a region, is known as outer billiard theory.

Many interesting problems can arise in the detailed study of billiards trajectories. For example, any smooth plane convex set has at least two double normals, so there are always two distinct "to and from" paths for any smoothly curved table. Analysis of billiards path can involve sophisticated use of ergodic theory and dynamical systems.

One can also consider billiard paths on polygonal billiard tables. The only closed billiard path of a single circuit in an acute triangle is the pedal triangle. There are an infinite number of multiple-circuit paths, but all segments are parallel to the sides of the pedal triangle. There exists a closed billiard path inside a cyclic quadrilateral if its circum centre lies inside the quadrilateral.

G D Birkhoff was first to consider billiards systematically as models for problems of classical mechanics. Birkhoff considered billiards only in smooth convex domains; he did not think about billiards in polygons, or in non-convex domains.

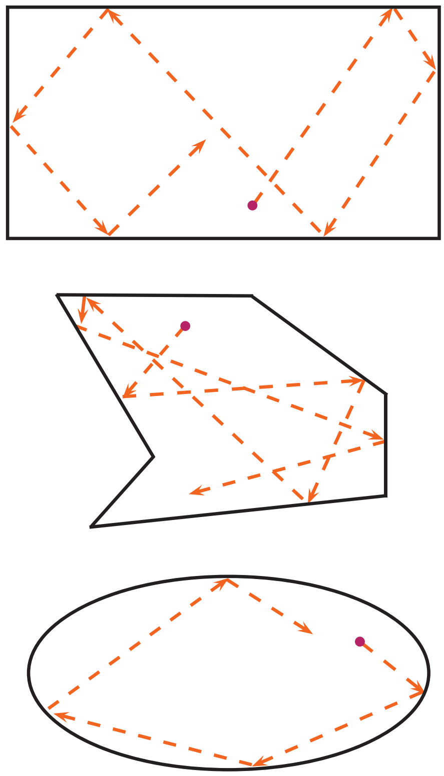

Fig. 1. Examples of billiard tables and trajectories

Mathematical theory of chaotic billiards was born in 1970 when Ya Sinai published his seminal paper [8]. During these years it grew and developed at a remarkable speed, and became a well-established and an important area within the modern theory of dynamical systems and statistical mechanics.

Now a mathematical billiard is a popular object of study: a MathSciNet and Google search shows that about 2000 publications devoted to billiards have appeared in mathematical and physical literatures over the years. These literatures include research papers as well as monographs, textbooks, and popular literature.

The book [2] is written in an accessible manner, and touch upon a broad variety of questions. This book can undoubtedly provide pleasurable and instructive reading for any mathematician or physicist interested in billiards, dynamical systems, ergodic theory, mechanics, geometry, partial differential equations and mathematical foundations of statistical mechanics. A shorter related "popularisation" text on the same subject is [3]. For an advanced, follow-up, graduate/research level book, refer to the book [6].

In [2] the authors use a large number of very nice and interesting problems from mathematics and physics to illustrate the multiple facets and applications of billiards. The main theme of the book is the study of the behaviour of billiard trajectories in various domains (for instance, existence of periodic trajectories), the relationships between this behaviour and the topology/geometry of the domain, and the conclusions that can be inferred from such results in geometry, mechanics and statistical physics. This book also contains some history of the game of billiards and development of the mathematical interest in billiards, with some unexpected problems designed to excite the interest of high school students: for instance, a billiard model can be used to solve some problems about measuring the amount of water in vessels (which we present in the next section). Moreover, the book [2] discusses the billiards in a disc and in an ellipse. Also the mechanical system "gas of absolutely rigid spheres" and its relations with billiards are studied. These problems lead to billiards in higher-dimensional space. Moreover, billiards in polygons and polyhedra are studied. In many cases the study of billiards in polygons reduces to the study of the trajectories of a point moving on a two-dimensional surface with two or more "holes". Such surfaces arise in classical mechanics when one considers problems connected with integrable and nearly integrable dynamical systems. The problem of the existence of periodic trajectories in polygons and polyhedra is studied.

An introduction to problems related with billiards for a more advanced reader can be found in Chapter 6 of [1]. The next level is represented in the book [10]. The book [4] contains a fairly detailed modern exposition of the theory of convex billiards and twist maps. A serious but still accessible exposition of the theory of parabolic billiards in its modern state is contained in a survey paper [7]. The volume [9] contains rich material on hyperbolic billiards and related questions.

Notable billiard tables are:

Hadamard's billiards. Hadamard's billiards concern the motion of a free point particle on a surface of constant negative curvature, in particular, the simplest compact Riemann surface with negative curvature, a surface of genus 2 (a two-holed donut). The model is exactly solvable, and is given by the geodesic flow on the surface. It is the earliest example of deterministic chaos ever studied, having been introduced by Jacques Hadamard in 1898.

Artin's billiards. Artin's billiards concern the free motion of a point particle on a surface of constant negative curvature, in particular, the simplest non-compact Riemann surface, a surface with one cusp. The billiards are notable in being exactly solvable, and being not only ergodic but also strongly mixing. Thus they are an example of an Anosov system. Artin billiards were first studied by Emil Artin in 1924.

Sinai's billiards. The table of the Sinai billiard is a square with a disk removed from its centre; the table is flat, having no curvature. The billiard arises from studying the behaviour of two interacting disks bouncing inside a square, reflecting off the boundaries of the square and off each other. By eliminating the centre of mass as a configuration variable, the dynamics of two interacting disks reduces to the dynamics in the Sinai billiard.

The billiard was introduced by Ya Sinai as an example of an interacting Hamiltonian system that displays physical thermodynamic properties: it is ergodic and has a positive Lyapunov exponent. As a model of a classical gas, the Sinai billiard is sometimes called the Lorentz gas.

Sinai's great achievement with this model was to show that the classical Boltzmann–Gibbs ensemble for an ideal gas is essentially the maximally chaotic Hadamard billiards.

For more historical notes on mathematical billiards see [5]. In the next sections we shall give some elementary applications of mathematical billiards.

2. Pouring Problems

Problem 1. *There are two vessels with capacities 7 and 11 litres and there is a greater of a flank filled with water. How to measure by these vessels exactly 1 litre of water*?

Solution. In the problem the billiard table can be considered as a parallelogram (see Fig. 2).

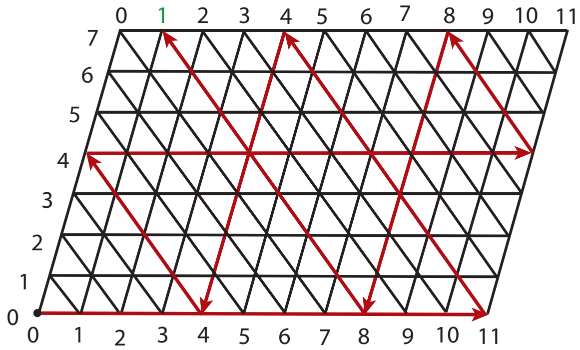

Fig. 2. The trajectory separating 1 litre of water

The sides of the table must be 7 and 11. Following the trajectory shown in Fig. 2 we can conclude the following:

1. The ball starts its trajectory at the point $(0,0)$ (the left bottom vertex). This position of the ball means that both vessels are empty.
2. In the next position it goes to $(0,11)$ this means the big vessel is full and the small vessel is empty.
3. Then it goes to the position $(7,4)$ which means that water has been poured from the large vessel to the small one.
4. The next position $(0,4)$ corresponds to the act that the small vessel has been poured out.

We should continue to follow the trajectory until one of the vessels will contain exactly 1 litre of water. The Fig. 2 shows that in the 8th step the large vessel contains exactly 1 litre. Then the described algorithm gives the solution of the problem.

Remark. If one first directs the ball to point $(7,0)$ (the left top vertex) then to get 1 litre, one has to do 25 steps. It is easy to check that by the mathematical billiard of Fig. 2 one can measure i litre of water for any $i = 1,2,\dots,11$. Just continue the trajectory until to a point with a coordinate equal to i.

Problem 2. *There is a vessel with capacity 8, which is full of water. There are two empty vessels with capacities 3 and 5 litres. How to pour the water in two greater vessels equally (i.e. both vessels must contain exactly 4 litres of water)?*

Solution. The table for this problem is a 3×5 parallelogram (see Fig. 3).

The large diagonal of the parallelogram, which corresponds to the vessel with capacity 8, is divided into 8 partes by the inclined straight lines. Following the trajectory, shown in Fig. 3 we should go until it is separated into 4 litres. The

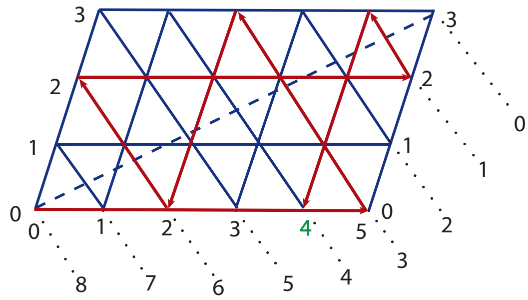

Fig. 3. The trajectory dividing 8 litres to two 4 litres

trajectory is

$$(0,0,8) \to (0,5,3) \to (3,2,3) \to (0,2,6) \to (2,0,6) \to (2,5,1) \to (3,4,1) \to (0,4,4).$$

This trajectory gives the algorithm of the solution.

Remark. If two smaller vessels have coprime (relatively prime) capacities (i.e. the capacity (volume) numbers do not have a common divisor $\neq 1$) and the biggest vessel has a capacity larger (or equal) than the sum of the capacities of the smaller vessels then using these three vessels one can measure water with litres: from 1 until the capacity of the mid vessel. For example, if there are three vessels with capacities 12, 13 and 26 respectively. Then one can measure l litre of water for any $l \in \{1,2,\dots,13\}$.

3. Billiard in the Circle

The circle enjoys rotational symmetry, and a billiard trajectory is completely determined by the angle α made with the circle. This angle remains the same after each reflection. Each consecutive impact point is obtained from the previous one by a circle rotation through angle $\theta = 2\alpha$.

If $\theta = \frac{2\pi p}{q}$, then every billiard orbit is q-periodic and makes p turns about the circle; one says that the *rotation number* of such an orbit is $\frac{p}{q}$. If θ is not a rational multiple of π, then every orbit is infinite. The first result on π-irrational rotations of the circle is due to Jacobi. Denote the circle rotation through angle θ by T_θ.

The following theorem is well known.

Theorem 1. *If θ is π-irrational, then the T_θ-orbit of every point is dense. In other words, every interval contains points of this orbit.*

Corollary. *If θ is π-irrational, then the T_θ-orbit has infinitely many points in any arc Δ of the circle.*

Let us study the sequence $x_n = x + n\theta \bmod 2\pi$ with π-irrational θ. If $\theta = \frac{2\pi p}{q}$, this sequence consists of q elements which are distributed in the circle very regularly. Should one expect a similar regular distribution for π-irrational θ?

The adequate notion is that of *equidistribution* (or *uniform* distribution). Given an arc I, let $k(n)$ be the number of terms in the sequence $x_0, \ldots, x_{n-1}$ that lie in I. The sequence is called equidistributed on the circle if

$$\lim_{n\to\infty} \frac{k(n)}{n} = \frac{|I|}{2\pi},$$

for every I.

The next theorem is due to Kronecker and Weyl; it implies Theorem 1.

Theorem 2. *If θ is π-irrational, then the sequence $x_n = x + n\theta \bmod 2\pi$ is equidistributed on the circle.*

Now we shall give some applications of Theorems 1 and 2.

Problem 3. *Distribution of first digits. Consider the sequence*

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$$

consisting of consecutive powers of 2.

Can a power of 2 start with 2012?

Is a term in this sequence more likely to start with 3 or 4?

Solution. Let us consider the second question: 2^n has the first digit k if, for some non-negative integer q, one has

$$k10^q \le 2^n < (k+1)10^q.$$

Take logarithm base 10:

$$\log k + q \le n \log 2 < \log(k+1) + q. \qquad (1)$$

Since q is of no concern to us, let us consider fractional parts of the numbers involved. Denote by $\{x\}$ the fractional part of the real number x. Inequalities (1) mean that $\{n \log 2\}$ belongs to the interval $I = [\log k, \log(k+1))$. Note that $\log 2$ is an irrational number. Thus by Theorem 1 there is a number n_0 such that $2^{n_0} = k\ldots$. Using Theorem 2, we obtain the following result.

Corollary. *The probability $p(k)$ for a power of 2 to start with digit k equals* $\log(k+1) - \log k$.

The values of these probabilities are approximately as follows:

$$p(1) = 0.301, \quad p(2) = 0.176, \quad p(3) = 0.125,$$
$$p(4) = 0.097, \quad p(5) = 0.079, \quad p(6) = 0.067,$$
$$p(7) = 0.058, \quad p(8) = 0.051, \quad p(9) = 0.046.$$

We see that $p(k)$ monotonically decreases with k; in particular, 1 is about 6 times as likely to be the first digit as 9.

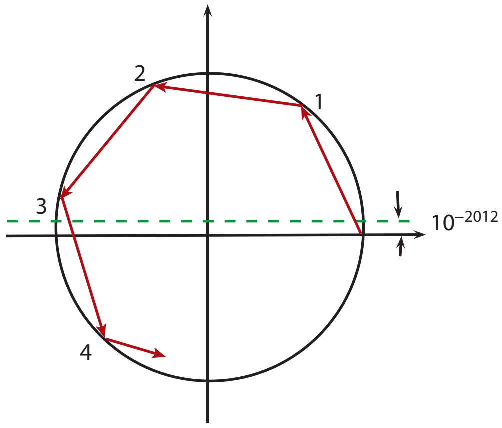

Fig. 4. Billiard in the circle

Exercise. (a) What is the distribution of the first digits in the sequence $2^n C$ where C is a constant?

(b) Find the probability that the first m digits of a power of 2 is a given combination $k_1 k_2 \ldots k_m$.

(c) Investigate similar questions for powers of other numbers.

(d) Prove that if p is such that $p \ne 10^q$ (for some $q = 1, 2, \ldots$) then the sequence $p, p^2, p^3, \ldots$ has a term with the first m digits is a given combination $k_1 k_2 \ldots k_m$.

Remark. Surprisingly, many "real life" sequences enjoy a similar distribution of first digits! This was first noted in 1881 in a 2-page article by American astronomer S Newcomb. This article opens as follows: "That the ten digits do not occur with equal frequency must be evident to anyone making much use of logarithmic tables, and noticing how much faster the first pages wear out than the last ones. The first significant figure is often 1 than any other digit, and the frequency diminishes up to 9."

Problem 4. *Is there a natural number n such that* $\sin n < 10^{-2012}$?

Solution. The answer is "exists"! To prove this consider a billiard on a circle with radius 1, which corresponds to the rotation number $\theta = 1$ radian (see Fig. 3). Then sequence $\sin 0, \sin 1, \sin 2, \ldots$ on $[-1, 1]$ corresponds to the trajectory $0, 1, 2, \ldots$ of the billiard with the starting point 0. Since 1 radian is π-irrational, by Theorem 1 we get the result. Note that the question is trivial if one considers $x \in R$ instead of $n = 1, 2, \ldots$.

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