Vicious Queues and Vicious Circles

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1. Introduction

> *We have to be willing to wrestle with paradox in pursuing understanding.*
>
> — Harold Evans

This article presents an introduction to some of the fascinating paradoxes that arise in logic and the foundations of mathematics. The study of paradoxes spans a recorded history of nearly two thousand six hundred years of intellectual studies in philosophy, mathematics, logic, and other disciplines. It is not surprising that the number of known paradoxes is quite large, and we can only touch upon a tiny fraction of them in this article. However, we try to discern a pattern even among the few paradoxes that we do present, and also point out several connections between paradoxes and the proofs of a number of important theorems in mathematics.

A bird's eye view of the paradoxes shows recurrent use of certain features such as self-reference, indexicals, negation, and circularity among others, and also the prevalence of certain patterns of reasoning. A closer look at the formulations of the paradoxes shows that some of them consciously restrict themselves, as it were, to the use of a smaller palette of such features and yet succeed in the formulation of the paradox. Such formulations have a greater impact than those formulations which are prodigal with basic features, since they are a clear demonstration of the fact that the absence of the certain features in the underlying theory is no guarantee of their eventual freedom from paradox.

The connections between paradoxes and proofs of mathematical theorems arises due to the fact that there are several similarities and common elements between the formulation and analysis of paradoxes on the one hand and *reductio ad absurdum* proofs of various theorems in mathematics on the other hand. We indicate, very briefly, some of the interesting connections that have been intensely explored by many other mathematicians.

2. Paradox and Circularity

> *How wonderful that we have met with a paradox. Now we have some hope of making progress.*
>
> — Niels Bohr

Paradoxes may be classified into two basic types — logical paradoxes and semantic paradoxes. Logical paradoxes involve notions only from the theory of sets. The most well known among the logical paradoxes is Russell's paradox [3] from Set theory:

The set of all sets that are not members of themselves is both a member of itself, and also not a member of itself.

Other examples of logical paradoxes are Cantor's paradox and Burali-Forti's paradox. Unlike logical paradoxes, the semantic paradoxes make use of notions which do not occur in the standard language of set theory. In this article, we will focus on some semantic paradoxes.

2.1. Grelling's paradox

There are some adjectives that describe themselves and some that do not. "English" is an English word; while "French" is not a French word. "Polysyllabic" is polysyllabic; but "monosyllabic" is not monosyllabic. Call all adjectives that describe themselves "autological". Call all adjectives that do not describe themselves "heterological". Grelling's paradox [3] is brought about by the question: *Is "heterological" heterological?* And the answer is: It is if only if it is not.

Grelling's paradox shows that the words "autological" and "heterological" do not form two well-defined categories into which all adjectives fall. It demonstrates that natural language does not necessarily partition all objects of thought into well-defined categories that will stand up to arbitrary logical scrutiny. Grelling's paradox can be translated into Russell's paradox. Identify each adjective with the set of objects to which that adjective applies. Thus an autological word is a set, one of whose elements is the set itself. The question of whether the word "heterological" is heterological becomes the question of whether the set of all sets not containing themselves contains itself as an element.

2.2. Epimenides's paradox

The Epimenides's paradox is also known as the Liar's paradox. The very first person known to contemplate the Liar's paradox [4] was the Greek philosopher Eubulides of Miletus who said "*A man says that he is lying. Is what he says true or false?*." One of the simplest ways of formulating the liar's paradox, is by the statement: *This sentence is false*.

It might seem at first that a sentence must directly refer to its own truth value, in order to construct a paradox. But this is not necessary. It is possible to avoid having a sentence directly refer to its own truth value, and still construct the paradox using a pair of sentences:
The following sentence is true.
The preceding sentence is false.
Neither of the above sentences refer to their own truth values, but together they construct the Liar's paradox.

The Liar's paradox, when it is removed from the domain of *truth* to that of *provability*, is closely associated to Gödel's Incompleteness Theorem [5].

2.3. Quine's paradox

Willard Quine [3] showed that it is possible to avoid direct self-reference, as embodied e.g. in the word "this", and yet construct the liar's paradox in a single sentence, as follows:
"yields falsehood when appended to its own quotation" yields falsehood when appended to its own quotation.

Quine's paradox is an algorithm for constructing yet another sentence. Say $X \equiv$ *yields falsehood when appended to its own quotation*. Then X's quotation $\equiv$ "*yields falsehood when appended to its own quotation*". X appended to X's quotation gives *"yields falsehood when appended to its own quotation" yields falsehood when appended to its own quotation.* In other words, the sentence says that it is false. The liar's paradox once again!

2.4. Berry's paradox

Berry's paradox was published in 1908 by Bertrand Russell [3]. It reads: *The smallest positive integer which to be specified requires more characters than there are in this sentence.*

Does this sentence specify a positive integer? The sentence has 114 characters (counting spaces between words and the period but not the quotation marks). Yet it supposedly specifies an integer that, by definition, requires more than 114 characters to be specified. This is clearly paradoxical.

It can also rewritten as: *The smallest positive integer that cannot be defined in less than fourteen words.* It is reasonable to assume that this is a specification for a number. There are a finite number of sentences of fewer than fourteen words, and some finite subset of them specify unique positive integers. So there is clearly some positive number that is the smallest integer not in that finite set. But the Berry sentence itself is a specification for that number in only thirteen words.

Berry's paradox is the starting point for the proof of a result in the area of algorithmic information theory, called Chaitin's theorem [1], which uses programs or proofs of bounded lengths to construct a rigorous version of this paradox. It is also closely related to the notions of Kolmogorov complexity and Martin-Löf randomness.

2.5. Löb's paradox

Löb's paradox [6] shows that it is possible to derive any arbitrary sentence from a sentence with self-reference and some apparently innocuous logical deduction rules.

Let A be any sentence. Let B be the sentence: "*If this sentence is true then A*". So, B asserts: "If B is true, then A".

Now consider the following argument: Assume B is true (hypothesis); then by B, since B is true, A holds. This argument shows that, if B is true, then A. But this is exactly what B asserts. Hence B is true. Therefore, by B, since B is true, A is true. Thus, every sentence is true.

Löb's paradox has close connections to Löb's theorem [6] in provability logic. Löb's theorem states that:

$$(PA \vdash Bew(\#P) \rightarrow P) \rightarrow (PA \vdash P)$$

where PA denotes Peano Arithmetic, $Bew(\#P)$ means that the formula with Gödel number P is provable, and the symbol $\vdash$ stands for provability. Löb's theorem in provability logic states that, in a theory with Peano arithmetic, for any formula P, if it is provable that "*if P is provable then P*", then "*P is provable*". Provability logic abstracts away from the details of encodings used in Gödel's incompleteness theorems. This is achieved by expressing the provability of ϕ in the given system in the language of modal logic, by means of the modality $\Box\phi$. Löb's theorem is formalised by the axiom:

$$\Box(\Box P \to P) \to \Box P$$

known as the Gödel–Löb axiom. It is sometimes also formalised by means of an inference rule that infers $\Box P$ from $\Box(\Box P \to P)$.

2.6. Curry's paradox

Haskell Curry [2], in 1942, was the first to show that the negation operator is not a necessary element in formulating Russell-type paradoxes. Curry's paradox refers to a family of paradoxes, which can be formulated in any language which satisfies various sets of conditions. One such set is as follows: The language must contain apparatus which lets it refer to, and talk about, its own sentences (such as quotation marks, names, or expressions like "this sentence"). The language must contain its own truth-predicate: that is, the language, call it "L", must contain a predicate meaning "true-in-L", and the ability to ascribe this predicate to any sentence.

Set theories which allow unrestricted comprehension satisfy the required conditions. In such set theories we can prove any logical statement Y from the set

$$X \equiv \{x | x \in x \to Y\}.$$

The proof proceeds as:

1. $X \in X \Leftrightarrow (X \in X \to Y)$ (Defn. of X)
2. $X \in X \to (X \in X \to Y)$ (from 1)
3. $X \in X \to Y$ (from 2 and contraction)
4. $(X \in X \to Y) \to X \in X$ (from 1)
5. $X \in X$ (from 3 and 4)
6. Y (from 3 and 5)

The term *contraction* is step 3 refers to a standard rule of inference, which says that from a statement of the form $P \to (P \to Q)$, we can infer $P \to Q$.

Note that unlike Russell's paradox, this paradox does not depend on what model of negation is used, as it is completely negation-free. There are various systems of logic such as paraconsistent logics which place restrictions on the use of the negation operator, thereby managing to avoid Russell's paradox easily. However, these systems still need to take care to avoid falling into Curry's paradox. The resolution of Curry's paradox is often a contentious issue because nontrivial resolutions are difficult and unintuitive.

2.7. Zwicker's paradox

William Zwicker formulated the Hypergame paradox [11] in the setting of game theory. As a preliminary to formulating the paradox, we need the notion of a finite game.

A two-person game may be defined to be *finite* if it satisfies the following conditions:

(i) Two players, A and B, move alternately, A going first. Each has complete knowledge of the other's moves.
(ii) There is no chance involved.
(iii) There are no ties, i.e. when a play of the game is complete, there is one winner.
(iv) Every play ends after finitely many moves.

We now define a two-person game called *Hypergame* with the following rules:

(i) On the first move, player A names any finite game F (called the subgame).
(ii) The players then proceed to play F, with B playing the role of A while F is being played.
(iii) The winner of the play of the subgame is declared to be the winner of the play of Hypergame.

Zwicker's Hypergame paradox is brought out by the question: Is Hypergame finite? As Hypergame satisfies the four conditions required for finite games, it is finite. If Hypergame is finite then player A can choose Hypergame as the finite game F of the first move. Now player B can name Hypergame as the first move. This process can lead to an infinite play, contrary to the assumption that Hypergame is finite.

2.8. Mirimanoff's paradox

This was formulated by Dmitri Mirimanoff in set theory [10]. It is also called the paradox of the class of all grounded classes.

A class X is said to be a *grounded class* when there is no infinite progression of classes X_1, X_2, ... (not necessarily all distinct) such that $\ldots \in X_2 \in X_1 \in X$.

Let Y be the class of all grounded classes. Mirimanoff's paradox is brought out by the question: Is Y itself grounded? Let us assume that Y itself is a grounded class. Hence $Y \in Y$ and so we have $\ldots Y \in Y \in Y \in Y$ contrary to groundedness of Y. Therefore Y is not a grounded class. If on the other hand Y is not grounded, then there is an infinite progression of classes $X_1, X_2, \ldots$ such that $\ldots \in X_2 \in X_1 \in Y$. Since $X_1 \in Y$, X_1 is a grounded class. But then $\ldots \in X_2 \in X_1$, which means X_1 in turn is not grounded, which is impossible since $X_1 \in Y$.

3. Paradox Without Circularity

> *If you try to fail, and succeed, which have you done?*
>
> — George Carlin

All known paradoxes in logic *seem* to require circularity in an unavoidable way. Each of them use either direct self-reference, or indirect loop-like self-reference. Yablo's paradox [8] was the first demonstration that self-reference is not a necessary condition for the construction of paradoxical sentences. Yablo's paradox is a non-self-referential Liar's paradox.

3.1. Yablo's paradox

Consider the following infinite sequence of sentences S_i where the indices "i, j, k" range over natural numbers:

$$(S_i) \quad : \quad \text{For all} \quad j > i, \; S_j \text{ is } \textit{false}\,.$$

Note that, in the above sequence of statements, each statement quantifies only over statements which occur later in the sequence. Now suppose S_k is true for some k. Then S_{k+1} is false, and so are all subsequent statements. As all subsequent statements are false, S_{k+1} is true, which is a contradiction. So S_k is false for all k. Looking at any particular i, this in turn means that S_i in fact holds, which is a contradiction.

It turns out that there are intimate connections between Yablo's paradox and generalisations Cantor's theorem in set theory, leading to alternate proofs of Cantor's theorem [7].

4. Conclusion

> *Such welcome and unwelcome things at once*
> *'Tis hard to reconcile.*
>
> — William Shakespeare

This article was meant to serve as a taster for the vast cornucopia of fascinating paradoxes [3] that arise in the study of logic and foundations of mathematics. Our listing of the paradoxes was certainly not meant to be exhaustive, and even the ones we mentioned are dealt with in much more detail and also in more unified ways in several other places in the literature [9].

Along with some of the paradoxes, we also briefly indicated connections with the proofs of some fundamental theorems. The close connections between paradoxes and proofs may indicate that the paradoxes could serve as springboards for the proofs of several other interesting theorems in mathematics.

Now that we have reached the concluding paragraph of our article devoted to paradoxes, it is time to explain the title. The phrase *vicious circle* is a standard term from logic which refers to a reasoning pattern where the hypothesis is used to prove the conclusion and the conclusion is in turn used to prove the hypothesis, and by this circularity often leads to a paradox. On the other hand, the phrase *vicious queue* was invented just for the purposes of this article. What could it possibly mean? Let the reader ponder.

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