Lovász's Umbrella*

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1. Foreword

The Hungarian mathematician László Lovász[1] shared with Elias M. Stein the 1999 Wolf Prize[2] in mathematics. The award citations list that "László Lovász has obtained ground-breaking results in discrete mathematics that have had significant applications to other areas of pure and applied mathematics as well as to theoretical computer science. He solved several outstanding problems, including the perfect graph conjecture, Kneser's conjecture, and the determination of the Shannon capacity of the pentagon, by introducing deep mathematical methods relying on geometric polyhedral and topological techniques. ..."

The purpose of this article is to introduce Lovász's exciting derivation for determining the Shannon capacity of the pentagon. Some issues that discrete mathematicians are concerned with in the 1960s shall also be touched. A brief introduction to graph theory is given first.

László Lovász

2. Origin of Graph Theory

While it is difficult to pinpoint the exact year when a subfield of mathematics starts, it is generally accepted that graph theory began with the paper [1] on the seven bridge problem by Euler in 1736.

In the 18th century, the city of Königsberg in Prussia (now Kaliningrad, Russia) was laid out on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges; see the schematic diagram on the left of Fig. 1. There was a puzzle to find a walk through the city that would cross each bridge once and only once.

After some trial and error, it is easy to see that this is impossible. Euler gave a more "precise" argument. First, Euler pointed out that the choice of a route inside each piece of land is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (thus laying the foundations of graph theory), eliminating all features except the list of the pieces of land and

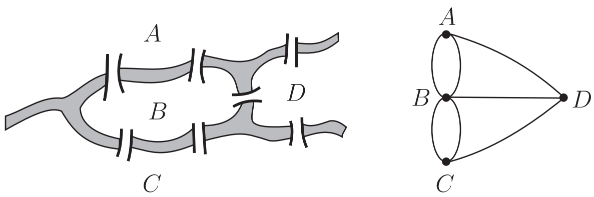

Fig. 1. (a) Diagram of 4 pieces of land connected by 7 bridges; (b) schematic representation of (a).

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[1]László Lovász (born March 9, 1948) is a Hungarian-American mathematician, best known for his work in combinatorics, for which he was awarded the Wolf Prize and the Knuth Prize in 1999, and the Kyoto Prize in 2010. In high school, Lovász won gold medals at the International Mathematical Olympiad (in years 1964, 1965, 1966 with two special prizes). Lovász received his Candidate of Sciences degree in 1970 at Hungarian Academy of Sciences. Lovász worked at the Eötvös University until 1975. Between 1975–1982, he led the Department of Geometry at the University of Szeged. In 1982, he returned to the Eötvös University, where he created the Department of Computer Science. Lovász was a professor at Yale University during the 1990s and was a collaborative member of the Microsoft Research Center until 2006. He returned to Eötvös Loránd University, Budapest, where he was the Director of the Mathematical Institute (2006–2011). He served as the President of the International Mathematical Union between January 1, 2007 and December 31, 2010.

[2]The Wolf Foundation began its activities in 1976, with an initial endowment fund of $10 million donated by Dr Ricardo Subiranay Lobo Wolf and his wife Francisca. It is awarded in six fields: Agriculture, Chemistry, Mathematics, Medicine, Physics, and an Arts prize that rotates between architecture, music, painting, and sculpture.

the bridges connecting them. In modern terms, one replaces each piece of land by an abstract "*vertex*" or node, and each bridge with an abstract connection, an "*edge*", which only serves to record which pair of vertices (pieces of land) is connected by that bridge. The resulting mathematical structure is called a *graph*; see Fig. 1(b).[3] The problem then is to start from some vertex and to traverse along an edge to another vertex, then to traverse along another edge to another vertex etc, until each edge has been traversed exactly once.

Euler considered not only the special case of seven bridges but also general graphs. For any vertex v of a graph, define its *degree* $d(v)$ to be the number of edges incident to it. As examples, in Fig. 1, $d(A) = d(C) = d(D) = 3$ and $d(B) = 5$. To sum up the degrees of all vertices, we label each edge incident to each vertex when add its degree. The degree sum is equal to the total number of labels made. As each edge has two end vertices, it is labelled exactly twice. Consequently, Euler established that

$$\sum_{v} d(v) = 2m,$$

where m is the number of edges in the graph. The above argument is known as *double counting*, which is a useful method. From the degree sum formula, we know that the number of vertices with odd degree is even. For instance, the graph in Fig. 1 has four vertices of odd degree.

The argument by Euler to prove that the seven bridge problem has no solution is as follows. Suppose that there is a feasible tour from x to y (which is called a *walk* in graph theory). For any vertex z, distinct from x and y, after entering z through an edge, one also needs to leave z from another edge. Hence the edges incident to z appear in pairs, and so $d(z)$ is even. However, the degrees of all vertices of the graph in Fig. 1 are odd, which is impossible.

The main part of Euler's paper was to discuss the converse. It explained that if the graph is connected and all vertices have even degrees, then one can start from any vertex to traverse a walk using each edge exactly once and finally come back to the original vertex.[4]

[3]This was first appeared in the book "Mathematical Recreations and Problems of Past and Present Times" by W. W. Rouse Ball (1892).

[4]In fact Euler's argument on this part was incomplete. However, it is not hard to fix it. People now still give the contribution to Euler and call such a walk an "Euler tour".

If a connected graph has $2k$ vertices of odd degree, by pairing them and adding k new edges connecting them will result in a connected graph whose vertices all have even degrees. Hence there is a walk from some vertex traversing each edge exactly once and returning to the original vertex. Deleting the k added edges, k walks of the original graph are produced. Hence, in the seven bridge problem, two walks are needed, rather than one.

3. First Book by König

During the two hundred years from 1736 to 1936, researchers in different areas studied the same concept of graph discovered by Euler, but using different terminologies in various contexts; see [2]. In 1936, König wrote the first book on graph theory [3]. Graph theory has been developing quickly since then. Its theory and applications are recognized in mathematics as well as many other fields. Related books increase exponentially.

For the convenience of our discussion, some terminologies are fixed. A *graph* is an ordered pair $G = (V, E)$, where V is a nonempty finite set whose elements are called *vertices*, and E is a set of some unordered pairs of distinct vertices called *edges*. Sometimes, $V(G)$ is used for the vertex set and $E(G)$ for the edge set of graph G. To simplify the notation, an edge $e = \{u, v\}$ is often written as uv such that uv is the same as vu. In this case, u and v are called the *end vertices* of e. It is also said that e and u (or v) are *incident*; also, u and v are *adjacent*, or u and v are *neighbors* to each other. Let $N(v)$ be the set of all neighbors of v.

A graph is often drawn explicitly for convenience of visualization. For instance, Fig. 2 shows a graph with $V = \{a, b, c, d, e\}$ and $E = \{ab, ae, bc, cd, de\}$. If the labels of the vertices are not important in the discussion, they may be omitted. One may only label those vertices whose names are relevant in the discussion.

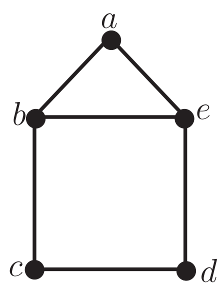

Fig. 2. A graph of 5 vertices and 6 edges.

There are variations on the definition of a graph. If more than one edge is allowed between two vertices, then one has *multigraphs* with *multiple edges*. If it is further allowed for an edge to have two identical end vertices, then one has *pseudographs* with *loops*. If a direction is assigned to each edge (in this case, $(u, v) = uv$ and $(v, u) = vu$ are regarded as distinct), then one has *directed graphs*. If $E(G)$ is allowed to be infinite, then one has *infinite graphs*.

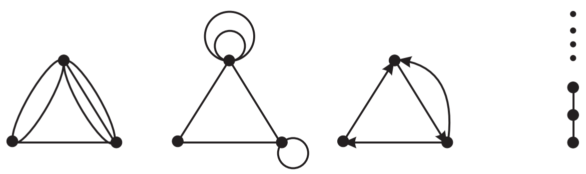

Fig. 3. Examples of a multigraph, a pseudograph, a directed graph and an infinite graph.

A *walk* in a graph is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$, where v_{i-1} and v_i are the end vertices of the edge e_i for $1 \le i \le k$. In the definition, v_0 is the *starting vertex*, v_k the *end vertex* and k the *length* of the walk. The walk is *closed* when $v_0 = v_k$, and is *open* when $v_0 \neq v_k$. If a graph has no multiple edges or loops, then e_i is determined by v_{i-1} and v_i. In this case, one may use $v_0, v_1, \ldots, v_k$ for a walk. A *trial* is a walk without repeated edges. A walk without repeated vertices is known as a path. An *Euler tour* is a closed trial in which every edge of the graph appears exactly once. When every two vertices of a graph have a walk between them, it is known as connected graph. In summary, Euler's theorem can be stated as follows.

Theorem (Euler). *For a graph G without vertices of degree zero, G has an Euler tour if and only if G is connected and every vertex has an even degree.*

A *bipartite graph* is a graph whose vertex set can be partitioned into A and B such that every edge has one end vertex in A and another end vertex in B. A *matching* in a graph is an edge set in which no two distinct edges have a same end vertex. The study of matchings in bipartite graphs was popular at the beginning of the 20th century. König was attracted to this problem, and he found graph theory interesting. As the result of his studies, he published the first book on graph theory in 1936 [3]. In this book, bipartite matching occupied an important part. The famous Hungarian algorithm was established by König and Egerváry to determine a maximum matching of a bipartite graph; see [4, 5].

4. Vertex Coloring of Graphs

Graph coloring has its origin in a problem posed by the British student Francis Guthrie (who later became a professor in mathematics in South Africa) who asked whether a plane graph can be face four-colorable. A face coloring of a plane graph is the same as a vertex coloring of its dual graph. The dual graph of a plane graph is a plane graph whose vertices correspond to the faces of the original graph and two vertices in the dual graph are adjacent if the corresponding faces share an edge. So people now often study vertex coloring for convenience. After the problem has studied for one century, many directions and tools together with exciting results on graph coloring are established. Finally, Appel, Haken and Koch [6, 7] proved the Four Color Theorem in 1977, by means of computers using the "discharging method". Their method relies heavily on computer and needs massive computing. The proof is not satisfying to all mathematicians. Even now, there are still people seeking a neat and "readable" proof for the theorem. A comment by them is that "A good mathematical proof should be like a poem, but this is just a telephone book." Although the proof is simplified by Robertsen, Sanders, Seymour and Thomas [8], computer aid is still unavoidable.

Our interest in graph coloring is not on the Four Color Theorem, instead is on the applications in the real world such as scheduling, time table, channel assignment, resource allocation, experimental design etc.

A *proper k-coloring* of a graph G is a mapping $f\colon V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices x and y. The *chromatic number* $\chi(G)$ of G is the minimum k for which G has a proper k-coloring. As we can properly color the vertices of the graph G in Fig. 2 by $f(a) = f(c) = 1$, $f(b) = f(d) = 2$, $f(e) = 3$, it is the case that $\chi(G) \le 3$. In fact, $\chi(G) = 3$ since a, b, e need different colors.

In a graph G, an *independent set* (respectively, *clique*) is a subset $S \subseteq V(G)$ in which every two distinct vertices are not adjacent (respectively, are adjacent). The *independence number* $\alpha(G)$ (respectively, *clique number* $\omega(G)$) of G is the maximum size of an independent set (respectively, clique). From the definition of a proper k-coloring f, it is the case that $f^{-1}(i) = \{x \in V(G)\colon f(x) = i\}$ is an independent set for $1 \le i \le k$. Consequently, the chromatic number $\chi(G)$ is the minimum number of independent sets one can partition $V(G)$ into, where each independent set $f^{-1}(i)$ is called a *color class*.

The reason for graph coloring to have wide applications is because in the real applications one often needs to partition objects into classes with certain properties. If the objects are viewed as vertices of a graph and two objects do not have a certain property are linked by an edge, then the problem is often reduced to a graph coloring problem. An example is given as follows.

A university has n courses and the ith course uses the time slot $[a_i, b_i]$. The duty of the Academic Affair Office is to schedule the courses by using a minimum number of classrooms subject to the constraint that two courses with overlap time slots cannot use the same classroom. One may consider the graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{v_iv_j\colon\ i \ne j, [a_i, b_i] \cap [a_j, b_j] \ne \emptyset\}$. Coloring two courses that can use the same classroom by the same color produces a proper coloring of the graph. Hence $\chi(G)$ is the minimum number of classrooms needed.

The graph defined above by using intervals in the real line is called an *interval graph*. Graph coloring for interval graphs is a popular subject in the 1960s.

5. Origin of Perfect Graph

The chromatic number of an interval graph can be obtained by the following method which is known as greedy algorithm, as minimality creterion is applied. Suppose that G is an interval graph in which vertex v_i coresponds to the interval $[a_i, b_i]$ in the real line for $1 \le i \le n$. For convenience, assume that $a_1 \le a_2 \le \cdots \le a_n$.

Greedy algorithms look for simple, easy-to-implement solutions to complex, multi-step problems by deciding which next step will provide the most obvious benefit.

A greedy algorithm is used to color the vertices of G: for i from 1 to n, color v_i one by one by "the minimum positive integer not used for v_j with $j < i$ and $[a_i, b_i] \cap [a_j, b_j] \ne \emptyset$".

It is evident that the coloring produced is proper. Suppose that a total of k colors are used. The following primal-dual approach is used to prove that $\chi(G) = k$. First, for any graph G the *weak dual inequality*

$$\omega(G) \le \chi(G)$$

holds, since distinct vertices in a clique need to be colored by different colors. Let us return to the coloring in the interval graph. Suppose vertex v_i is colored by k. The reason it is colored by k is because that there are $j_1, j_2, \ldots, j_{k-1}$ (all less than i) such that v_{j_r} is colored by r $(1 \le r \le k-1)$ and $[a_{j_r}, b_{j_r}] \cap [a_i, b_i] \ne \emptyset$. However, $j_r < i$ implies that $a_{j_r} \le a_i$ and so $[a_{j_r}, b_{j_r}]$ contains a_i. These give that $\{v_{j_1}, v_{j_2}, \ldots, v_{j_{k-1}}, v_i\}$ is a clique of k vertices. Consequently,

$$k \le \omega(G) \le \chi(G) \le k,$$

and so $\omega(G) = \chi(G) = k$. The good result above is due to the fact that the interval graph G has the "perfect" property of $\omega(G) = \chi(G)$. This also appears in the work by Shannon to be described below. However, Berge (in the 1960s) defined the perfection of a graph requiring more conditions. A graph is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. An *induced subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) = \{xy \in E(G)\colon x, y \in V(H)\}$. The reason that Berge defined the perfection in this way is because that even when $\omega(G) = \chi(G)$, the graph may still contain an induced subgraph with very bad structure. For instance, no matter what graph H is, if it has n vertices, then for $G = H \cup K_n$ one always has $\omega(G) = \chi(G) = n$.

During the earlier years, besides the pair of graph parameters ω and χ, people also considered α and θ, where $\theta(G)$ is the minimum number of cliques required to partition the vertex set $V(G)$. As an independent set (respectively, a clique) in G is a clique (respectively, an independent set) in the complement G^c of G,

$$\alpha(G) = \omega(G^c) \text{ and } \theta(G) = \chi(G^c)$$

or equivalently

$$\omega(G) = \alpha(G^c) \text{ and } \chi(G) = \theta(G^c).$$

Hence, studying ω and χ for G is the same as studying α and θ for the complement graph G^c. Shannon's work is described below in terms of α and θ.

In 1956, Shannon [9] studied zero-fault information transmission. The aim is to transmit information without confusion. Suppose that there is a set V of "letters" to be used for transmitting messages. Construct a graph G with vertex set V and edge set $E = \{xy\colon x \text{ and } y \text{ are not confusable}\}$. Two messages $x_1x_2\cdots x_m$ and $y_1y_2\cdots y_m$ of length m are not *confusable* if $x_i \neq y_i$ and x_i and y_i are not confusable for some i. The aim is, for a fixed m, to find a maximum sized set of pairwise nonconfusable messages of length m. A simple method is to find an independent set S of size $\alpha(G)$. Then

$$\{x_1x_2\cdots x_m\colon \text{ each } x_i \in S\}$$

is such a message set of size $\alpha(G)^m$. The problem is whether a better answer exists.

The answer is positive. One may do better for some cases. Consider the example of $G = C_5$, the pentagon, with $V(C_5) = \{a, b, c, d, e\}$ and $E(C_5) = \{ab, bc, cd, de, ea\}$. It is easy to see that $\alpha(C_5) = 2$ and so there is a set of nonconfusable set of size 2^m. However, one may consider the set of $x_1x_2\cdots x_m$ such that for each odd i the pair x_ix_{i+1} is chosen from $\{aa, bc, ce, db, ed\}$. This produces a nonconfusable set of size $5^{\lfloor m/2\rfloor}$, which is significantly larger than 2^m when m is large.

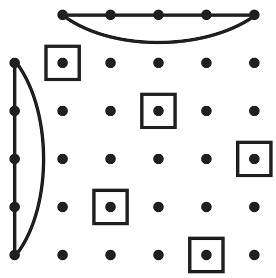

Fig. 4. Five choices of x_ix_{i+1} from $\{aa, bc, ce, db, ed\}$.

To simplify the description, the concept of strong product operation on graphs is introduced. The *strong product* of two graphs G and H is the graph $G\otimes H$ with vertex set $V(G\otimes H) = V(G)\times V(H)$ and edge set

$$E(G\otimes H) = \{(x, y)(x', y')\colon (x = x' \text{ or } xx' \in E(G)) \text{ and } (y = y' \text{ or } yy' \in E(H))\}.$$

For a positive integer m, let $G^m = G\otimes G\otimes\cdots\otimes G$ (m times). The number asked by Shannon is then the value $\alpha(G^m)$. Notice that for any two graph G and H,

$$\alpha(G\otimes H) \geq \alpha(G)\otimes(H)$$

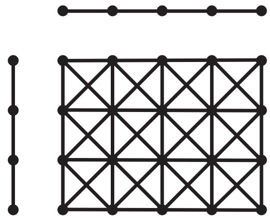

Fig. 5. $P_4 \otimes P_5$.

and so $\alpha(G^m) \geq \alpha(G)^m$. For $G = C_5$, one has $\alpha(C_5^m) \geq \alpha(C_5)^m = 2^m$. It was in fact obtained that $\alpha(C_5^m) \geq 5^{\lfloor m/2\rfloor}$. In general, $\alpha(G^m)$ grows exponentially. For convenience, one considers $\alpha(G^m)^{1/m}$, whose limit exists[5] as $m \to \infty$. Let $\psi(G) = \lim_{m\to\infty} \alpha(G^m)^{1/m}$, which is known as the *Shannon capacity*. Let us come back to C_5. What is $\psi(C_5)$? It was already known that $\psi(C_5) \geq \sqrt{5}$.

Before giving Lovász's proof for $\psi(C_5) = \sqrt{5}$, let us return to perfect graphs. First, similar to the weak perfect graph inequality $\chi(G) \geq \omega(G)$, one also has $\theta(G) \geq \alpha(G)$. One shows that

$$\alpha(G)\alpha(H) \leq \alpha(G\otimes H) \leq \theta(G\otimes H) \leq \theta(G)\theta(H).$$

Consequently, $\alpha(G)^m \leq \alpha(G^m) \leq \theta(G^m) \leq \theta(G)^m$ and so $\alpha(G) \leq \alpha(G^m)^{1/m} \leq \theta(G^m)^{1/m} \leq \theta(G)$. If $\alpha(G) = \theta(G)$, then the nice conclusion that $\psi(G) = \alpha(G)$ is obtained. This was why Berge imposed the condition for G to have $\alpha(G) = \theta(G)$. As mentioned before, it is easier to characterize graphs G for which $\alpha(H) = \theta(H)$ holds for all induced subgraphs H.

Since the relation between ω and χ of a graph G is the same as the relation between α and θ of its complement G^c, one in fact studies the perfection for both G and G^c. The first minimal nonperfect graph is C_5. This is why one has that $\psi(C_5) \geq \sqrt{5} > 2 = \alpha(C_5)$. When studying perfect graphs, Berge observed that if one of G and G^c is perfect, then so is the other. Interval graphs are examples with this nice property. It is easy to see that a perfect graph cannot contain C_{2k+1} and C_{2k+1}^c ($k \geq 2$) as induced subgraphs. The converse seems to be true as well. Hence he gave the following two famous conjectures.

(C1) Graph G is perfect if and only if G^c is perfect.

(C2) Graph G is perfect if and only if G does not contain C_{2k+1} and C_{2k+1}^c ($k \geq 2$) as induced subgraphs.

[5] In general, if a function $f\colon \mathbb{N} \to \mathbb{R}^+$ satisfying $f(m + n) \geq f(m)f(n)$ for any $m, n \in \mathbb{N}$, then $\lim_{m\to\infty} f(m)^{1/m}$ exists. This is the famous Fekete Theorem.

Notice that (C2) implies (C1). Hence, (C1) is known as the Weak Perfect Graph Conjecture and (C2) the Strong Perfect Graph Graph Conjecture. The Weak Perfect Graph Conjecture was proved by Lovász [7, 8] in 1972. This is also contained in the citation for Lovász to receive his Wolf Prize. The Strong Perfect Graph Conjecture was proved by Chudnovsky, Robertson, Seymour and Thomas [9] in 2006.

6. Lovász's Ingenuity

Finally, let us explain how Lovász determined the value of $\psi(C_5)$. The *tensor product* of two vectors $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$ is the vector

$$x \circ v = (x_1v_1, x_2v_1, \ldots, x_nv_1, x_1v_2, x_2v_2, \ldots, x_nv_2, \ldots, x_1v_m, x_2v_m, \ldots, x_nv_m)$$

in $\mathbb{R}^{nm}$. The *inner product* of two vectors x and y in $\mathbb{R}^m$ is $\langle x, y\rangle = \sum_{i=1}^m x_iy_i$. For any vectors x and y in $\mathbb{R}^n$ and v and w in $\mathbb{R}^m$,

$$\langle x \circ v, y \circ w\rangle = \langle x, y\rangle\langle v, w\rangle. \qquad (*)$$

Suppose that graph G has vertex set $V(G) = \{1, 2, \ldots, n\}$. A *standard orthonormal representation* of G is a family of unit vectors $(v^1, v^2, \ldots, v^n)$ in an Euclidean space such that v^i is orthogonal to v^j if i is not adjacent to j. Such a representation always exists, for instance a family of pairwise orthogonal unit vectors is a desired one. By formula $(*)$, one has:

Lemma 1. *If $(u^1, u^2, \ldots, u^n)$ and $(v^1, v^2, \ldots, v^n)$ are standard orthonormal representations of G and H respectively, then all $u^i \circ v^j$ form a standard orthonormal representation of $G \otimes H$.*

Define the *value* of a standard orthonormal representations $(u^1, u^2, \ldots, u^n)$ as

$$\min_c \max_{1\le i\le n} \frac{1}{\langle c, u^i\rangle^2},$$

where c runs over all unit vectors. A unit vector c attaining the minimum is called a *handle* of the representation. The notation $\phi(G)$ is used for the minimum value of a standard orthonormal representation of G, and call the representation attaining this value an *optimal representation*.

Lemma 2. $\phi(G \otimes H) \le \phi(G)\phi(H)$.

Proof. Suppose $(u^1, u^2, \ldots, u^n)$ and $(v^1, v^2, \ldots, v^n)$ are optimal representations of G and H with handles c and d, respectively. By formula $(*)$, $c \circ d$ is a unit vector. By $(*)$ again,

$$\begin{aligned}\phi(G \otimes H) &\le \max_{i,j} \frac{1}{\langle c \circ d, u^i \circ v^j\rangle^2} = \max_{i,j} \frac{1}{\langle c, u^i\rangle^2\langle d, v^j\rangle^2} \\ &= \phi(G)\phi(H).\end{aligned}$$ □

Lemma 3. $\psi(G) \le \phi(G)$.

Proof. First we prove that $\alpha(G) \le \phi(G)$. Suppose that $(u^1, u^2, \ldots, u^n)$ is an optimal representation of G with a handle c. Without loss of generality, assume that $\{1, 2, \ldots, k\}$ is a maximum independent set of G. Since $u^1, u^2, \ldots, u^k$ are pairwise orthogonal,

$$1 = \|c\|^2 \ge \sum_{i=1}^k \langle c, u^i\rangle^2 \ge \frac{\alpha(G)}{\phi(G)}.$$

This, together with Lemma 2, implies that $\alpha(G^n) \le \phi(G^n) \le \phi(G)^n$. Taking $(1/n)$th power and taking the limit give the lemma. □

Having the lemmas above at our disposal, we are now ready to give the wonderful proof of $\psi(C_5) = \sqrt{5}$ by Lovász. Consider an umbrella, as one in the real life, with the handle and the five bones of unit length. Assume the top of the umbrella is at the origin $O(0, 0, 0)$ of the 3-dimensional space $\mathbb{R}^3$. Another end of the handle is at $(0, 0, 1)$. The other ends A_1, A_2, A_3, A_4, A_5 of the five bones are also at $(0, 0, 1)$ when the umbrella is closed. When the umbrella is open, A_1, A_2, A_3, A_4, A_5 form a pentagon lying on the circle with center $(0, 0, \sqrt{1-r^2})$ and radius r in the plane $z = \sqrt{1-r^2}$ of the 3-dimensional space. The radius r increase from 0 to 1 during the opening of the umbrella. For the radius r, the length of each side of the pentagon is $2r\sin 36°$, and the distance between nonconsecutive A_i and A_j (such as A_1 and A_3) is $2r\sin 72°$. So the lengths of the sides of the isosceles triangle formed by OA_i and OA_j are $1, 1, 2r\sin 72°$. By the Pythagorean theorem, the angle between OA_i and OA_j is 90° when $2r\sin 72° = \sqrt{2}$. That is, when the umbrella is open to $r = \csc 72°/\sqrt{2} \approx 0.7435$, the angle between the two nonconsecutive bones is a right angle. The handle of the umbrella is used as vector c, and the bones as u^1, u^2, u^3, u^4, u^5 to form a standard orthonormal representation of C_5. Since $\langle c, u^i\rangle = 5^{-1/4}$, it is the case that $\psi(C_5) \le \phi(C_5) \le \sqrt{5}$. Consequently, $\psi(C_5) = \sqrt{5}$.

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