An Elementary Introduction to Categorical Representation Theory

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1. What is Categorification?

In this expository article, we will give a brief introduction to the categorical representation theory, some of which can be found in [8, 9]. The idea of categorification dates back to I. B. Frenkel. He proposed that one can construct a tensor category whose Grothendieck ring is isomorphic to the representation theory of the simplest quantum group $U_q(\mathfrak{sl}_2)$ (see, for example, [6, 7]). Since then, a construction of a tensor category or a 2-category whose Grothendieck group is isomorphic to a given algebraic structure has been referred to as *categorification*.

One of the most prominent examples of categorification is the Lascoux–Leclerc–Thibon–Ariki theory, which clarifies the mysterious connection between the representation theory of quantum affine algebras of type $A^{(1)}_{n-1}$ and the modular representation theory of Hecke algebras at roots of unity. In [18], Misra and Miwa constructed an integrable representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$, called the Fock space representation, on the space spanned by coloured Young diagrams. They also showed that the affine crystal consisting of n-reduced coloured Young diagrams is isomorphic to the highest weight crystal $B(\Lambda_0)$. In [16], using the Misra–Miwa construction, Lascoux, Leclerc and Thibon discovered an algorithm of computing Kashiwara's lower global basis (=Lusztig's canonical basis) elements corresponding to n-reduced Young diagrams and conjectured that the transition matrices evaluated at 1 coincide with the composition multiplicities of simple modules inside Specht modules over Hecke algebras.

The Lascoux–Leclerc–Thibon conjecture was proved by Ariki in more general form [1]. Combining the geometric method of Kazhdan–Lusztig and Ginzburg with combinatorics of Young diagrams and Young tableaux, Ariki proved that the complexified Grothendieck ring of finitely generated projective modules over cyclotomic Hecke algebras give a categorification of integrable highest weight modules over affine Kac–Moody algebras of type $A^{(1)}_{n-1}$. Moreover, he showed that the lower global basis is mapped onto the isomorphism classes of projective indecomposable modules, from which the Lascoux–Leclerc–Thibon conjecture follows. Actually, he proved the conjecture in more general form because his categorification theorem holds for highest weight modules of arbitrary level.

2. Khovanov–Lauda–Rouquier Algebras

Next problem is to prove a quantum version or a graded version of Ariki's categorification theorem. The key to this problem was discovered by Khovanov–Lauda and Rouquier. In [14, 15, 19], Khovanov–Lauda and Rouquier introduced a new family of graded algebras, called the *Khovanov–Lauda–Rouquier algebras* or *quiver Hecke algebras*, and proved that they provide a categorification of quantum groups associated with symmetrisable Cartan data.

Let $U_q(\mathfrak{g})$ be the quantum group associated with a symmetrisable Cartan datum and let $R = \bigoplus_{\beta\in Q^+} R(\beta)$ be the corresponding quiver Hecke algebra. Then it was shown in [14, 15, 19] that there exists an algebra isomorphism

$$U^-_{\mathbb{A}}(\mathfrak{g}) \simeq [\mathrm{Proj}(R)] = \bigoplus_{\beta\in Q^+}[\mathrm{Proj}(R(\beta))],$$

where $U^-_{\mathbb{A}}(\mathfrak{g})$ is the integral form of $U^-_q(\mathfrak{g})$ with $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$, and $[\mathrm{Proj}(R)]$ is the Grothendieck group of finitely generated graded projective R-modules. When the generalised Cartan matrix is symmetric, Varagnolo and Vasserot proved that the lower global basis corresponds to the isomorphism classes of projective indecomposable R-modules under this isomorphism [20].

Moreover, Khovanov and Lauda conjectured that the cyclotomic quiver Hecke algebras give a graded version of Ariki's categorification theorem in much more generality. For each dominant integral weight $\Lambda \in P^+$, the algebra R has a special quotient $R^\Lambda = \bigoplus_{\beta\in Q^+} R^\Lambda(\beta)$ which is called the *cyclotomic quiver Hecke algebra*. In [15], Khovanov and Lauda conjectured that $[\mathrm{Proj}(R^\Lambda)]$ has a $U_{\mathbb{A}}(\mathfrak{g})$-module structure and that there exists a $U_{\mathbb{A}}(\mathfrak{g})$-module isomorphism

$$V_{\mathbb{A}}(\Lambda) \simeq [\mathrm{Proj}(R^\Lambda)] = \bigoplus_{\beta\in Q^+}[\mathrm{Proj}(R^\Lambda(\beta))],$$

where $V_{\mathbb{A}}(\Lambda)$ denotes the $U_{\mathbb{A}}(\mathfrak{g})$-module generated by the highest weight vector v_Λ. It is called the *cyclotomic categorification conjecture*.

In [4], Brundan and Stroppel proved a special case of this conjecture in finite type A. In [3], Brundan and Kleshchev proved this conjecture for type A_∞ and $A_n^{(1)}$ using the isomorphism between R^Λ and the cyclotomic Hecke algebra H^Λ which was constructed in [2]. In [17], the crystal version of this conjecture was proved for all symmetrisable Kac–Moody algebras. That is, in [17], Lauda and Vazirani investigated the crystal structure on the set of isomorphism classes of irreducible graded modules over R and R^Λ, and showed that these crystals are isomorphic to the crystals $B(\infty)$ and $B(\Lambda)$, respectively.

3. Cyclotomic Categorification Theorem

This conjecture was proved by Kang and Kashiwara [8] for *all* symmetrisable quantum Kac–Moody algebras. Their proof is based on: (i) a detailed analysis of the structure of quiver Hecke algebras and their cyclotomic quotients, (ii) the proof of the exactness of restriction and induction functors, (iii) the existence of natural isomorphisms, which is the $\mathfrak{sl}_2$ categorification developed by Chuang and Rouquier [5].

The *cyclotomic categorification theorem* can be explained as follows. For each $i \in I$, let us consider the restriction functor and the induction functor:

$$E_i^\Lambda : \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \longrightarrow \mathrm{Mod}(R^\Lambda(\beta)),$$
$$F_i^\Lambda : \mathrm{Mod}(R^\Lambda(\beta)) \longrightarrow \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$$

defined by

$$E_i^\Lambda(N) = e(\beta, i)N,$$
$$F_i^\Lambda(M) = R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)} M,$$

where $M \in \mathrm{Mod}(R^\Lambda(\beta))$, $N \in \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$.

The first main theorem is that $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ is a projective right $R^\Lambda(\beta)$-module and $e(\beta,i)R^\Lambda(\beta+\alpha_i)$ is a projective left $R^\Lambda(\beta)$-module. Hence the functors E_i^Λ and F_i^Λ are exact and send projectives to projectives.

The second main theorem is the following: let $\lambda = \Lambda - \beta$.

(1) If $\langle h_i, \lambda\rangle \geqslant 0$, there exists a natural isomorphism of endofunctors on $\mathrm{Mod}(R^\Lambda(\beta))$:

$$q_i^{-2}F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i,\lambda\rangle-1} q_i^{2k}\mathrm{Id} \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda.$$

(2) If $\langle h_i, \lambda\rangle \leqslant 0$, there exists a natural isomorphism of endofunctors on $\mathrm{Mod}(R^\Lambda(\beta))$:

$$q_i^{-2}F_i^\Lambda E_i^\Lambda \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda \oplus \bigoplus_{k=0}^{-\langle h_i,\lambda\rangle-1} q_i^{2k-2}\mathrm{Id}.$$

Here, $q_i := q^{(\alpha_i,\alpha_i)/2}$ denotes the grade-shift functor. This is one of the axioms of the categorification of $U_q(\mathfrak{g})$ due to Rouquier [19]. We write $[\mathrm{Rep}(R^\Lambda)]$ for the Grothendieck group of the abelian category $\mathrm{Rep}(R^\Lambda)$ of R^Λ-modules that are finite-dimensional over the base field. It follows that the functors E_i^Λ, F_i^Λ $(i \in I)$ satisfy the mixed relations given above, and hence by [12, Proposition B.1], the Grothendieck groups $[\mathrm{Proj}(R^\Lambda)]$ and $[\mathrm{Rep}(R^\Lambda)]$ become integrable $U_q(\mathfrak{g})$-modules. Therefore, we obtain the categorification of the irreducible highest weight module $V(\Lambda)$:

$$[\mathrm{Proj}(R^\Lambda)] \simeq V_{\mathbb{A}}(\Lambda) \quad \text{and} \quad [\mathrm{Rep}(R^\Lambda)] \simeq V_{\mathbb{A}}(\Lambda)^\vee,$$

where $V_{\mathbb{A}}(\Lambda)^\vee$ is the dual of $V_{\mathbb{A}}(\Lambda)$ with respect to a non-degenerate symmetric bilinear form on $V(\Lambda)$. In other words, we obtain an integrable 2-representation of a 2-Kac–Moody algebra in the sense of Rouquier [19, Definition 5.1].

4. What Could Be Done Next?

The first possible application of the (cyclotomic) categorification theorem that comes up to our mind is the generalisation of the Brundan–Kleschchev graded decomposition number theorem [3] to all classical quantum affine algebras. For this purpose, we propose to go through the following steps.

(i) To find an algorithm of computing the lower global basis elements corresponding to proper Young walls. (For reduced Young walls, this algorithm was discovered by Kang and Kwon [13].)

(ii) To define graded modules or direct sum of virtual graded modules over quiver Hecke algebras that generalise the Specht modules over cyclotomic Hecke algebras.

(iii) To prove that the transition matrices defined in (i) give graded decomposition numbers of simple modules in the graded modules defined in (ii).

As was conjectured in [14] and shown in [20], when the Cartan datum is symmetric, the projective indecomposable modules over quiver Hecke algebras correspond to lower global basis of $U_q^-(\mathfrak{g})$ under the isomorphism given in the Khovanov–Lauda categorification theorem. By the Kang–Kashiwara cyclotomic categorification theorem, we obtain the cyclotomic version of the above correspondence.

But when the Cartan datum is not symmetric, the correspondence becomes much more subtle and complicated. Hence one can naturally ask the following questions:

(i) To which basis do the projective indecomposable modules correspond?

(ii) To which modules does the lower global basis correspond?

(iii) More generally, can we find a characterisation of a family of graded modules over quiver Hecke algebras and a perfect basis of $U_q^-(\mathfrak{g})$ from which we obtain a canonical 1-1 correspondence?

We would like to emphasise that there are many other exciting open problems and forthcoming developments in this area. We regret we could cover only a small part of them in this article due to the lack of time and space.

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