Topological Fixed Point Theory

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1. Introduction

Fixed point theory is very simple, but is based on fundamentals in Mathematics. For any continuous function $f : X \to X$, a fixed point of f is a point $x \in X$ satisfying the identity $f(x) = x$. Two fundamental theorems concerning fixed points are Banach Theorem and Brouwer Theorem. Banach theorem states that if X is a complete metric space and f is a contraction then f has a unique fixed point. In Brouwer theorem, X must be the closed unit ball in a Euclidean space. Then any f has a fixed point. But in this case, the set of fixed points is not necessarily a one-point set.

In Banach theorem, a metric on X is used in the crucial assumption that f is a contraction. The unit ball in a Euclidean space is also a metric space and the metric topology determines the continuity of continuous functions, however the essence of Brouwer theorem is a topological property of the unit ball, namely the unit ball is compact and contractible. Banach theorem and Brouwer theorem tell us a difference between two major branches of fixed point theory, metric space fixed point theory and topological fixed point theory. It is impossible to distinguish two fixed point theories in an exact way, and it is not easy to determine a certain topics belong to which branch. In general, the fixed point theory is regarded as a branch of topology. But due to deep influence on topics related to nonlinear analysis or dynamic systems, many parts of the fixed point theory can be considered as a branch of analysis.

In this article, we intend to introduce briefly topological fixed point theory. After dinner tonight, let us drink a cup of coffee. Stirring the coffee in the cup with a tea spoon, we have a look at the movement of the top of the coffee. Probably we can feel that there are points (or parts) of the coffee not moving at all. According to Brouwer theorem, such points always exist. More precisely speaking, we can regard this movement as a continuous function f on the closed disc D. Then by Brouwer theorem, there is a point p such that $f(p) = p$, namely f has a fixed point p in D.

Even though we can observe points of the coffee not moving, it is hard to grasp the position of such non-moving points. Brouwer theorem has nothing to do with the position of fixed points. Changing the way of stirring the coffee, fixed points may occur on the edge of the coffee and in this case, every part of the coffee you look at is moving because the fixed points occurred in the place where the coffee touches the cup ([4]).

2. Brief History

In 1880s, H Poincaré introduced topological methods in studying nonlinear problems of analysis. One of main ideas was to use fixed point results. Together with the study under the topological setting derived from Poincaré's analysis motivation, L E J Brouwer's fixed point theorem came into the world. In 1920s, the fixed point theory became a major branch of topology and afterwards it consistently became a major theme of the research.

S Lefschetz used homology theory in the study of fixed point theory and H Hopf developed it further and generalised successfully Brouwer's results. Lefschetz introduced the Lefschetz number $L(f)$ of continuous functions f. A continuous function $f : X \to X$ induces a homomorphism $f_* : H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ on the homology groups of X, and $L(f)$ is defined as follows:

$$L(f) = \sum_i (-1)^i \text{trace}\left\{ f_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q}) \right\}.$$

If this number is not equal to 0, then it is proven that f has a fixed point. Because the Lefschetz number is defined using the homology homomorphism, it is a homotopy invariant. Consequently, if the Lefschetz number $L(f)$ is nonzero, every continuous function homotopic to f has a fixed point.

In 1920s, J Nielsen concentrated on not only the existence problem of fixed points but also the problem of finding the (minimal) number of fixed points in the homotopy classes in a series of papers. The Nielsen theory is to study this homotopy invariant nowadays known as the Nielsen

number $N(f)$. The Nielsen number is defined in a very geometrical way. We say that two fixed point x and y of $f : X \to X$ are related when there is a path ω in X connecting x and y so that two paths ω and $f \circ \omega = f(\omega)$ are homotopic relative to the end points. In other words, it is the case when so-called a Nielsen disk exists. This relation defines an equivalence relation on the set of all fixed points of f, and the equivalence classes are called fixed point classes. Thus the set of fixed points is the union of the fixed point classes. To each fixed point class of f we can associate the index. When a fixed point class consists of finitely many points, its index is the sum of the winding numbers at each point of the fixed point class. Now when the fixed point class has a nonzero index, we call it essential. The number of essential fixed point classes is the very Nielsen number $N(f)$.

In 1930–40s, even though the fixed point theory was not a main stream of topology, there were important progress inside the fixed point theory itself. K Reidemeister and his student F Wecken introduced the Reidemeister trace, which combine the notion of the Lefschetz number and the Nielsen number. When the space X has the universal covering space $\tilde{X}$ and G is the group of all covering transformations, a continuous function $f : X \to X$ with a fixed lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ induces a group homomorphism $\varphi : G \to G$ as follows:

$$\varphi(\alpha)\tilde{f} = \tilde{f}\alpha, \quad \forall \alpha \in G.$$

In this case, φ defines an equivalence on G. Namely, $\alpha \sim \beta\alpha\varphi(\beta)^{-1}$ $(\alpha, \beta \in G)$. The equivalence classes are called Reidemeister classes $[\alpha]$ and we denote the set of all Reidemeister classes by G_φ, and we can consider the free Abelian group $\mathbb{Z}G_\varphi$ with generators the Reidemeister classes. An important consequence is that there is a one-to-one correspondence between the Nielsen's fixed point classes and the Reidemeister classes. Thus the Nielsen classes $\mathbb{F}$ can be indexed by the Reidemeister classes $[\alpha]$; $\mathbb{F} = \mathbb{F}_\alpha$. The Reidemeister number $R(\varphi)$ is the number of the Reidemeister classes. The Lefschetz $L(f)$ defined using the homology groups is the sum of local indices, which is an important result obtained by Lefschetz. That is,

$$L(f) = \sum_{[\alpha]} \mathrm{index}(f, \mathbb{F}_\alpha) \in \mathbb{Z}.$$

On the other hand, the Reidemeister trace of f is defined as follows:

$$R(f) = \sum_{[\alpha]} \mathrm{index}(f, \mathbb{F}_\alpha)[\alpha] \in \mathbb{Z}G_\varphi.$$

The number of nonzero terms in this sum is the Nielsen number. In particular, Wecken proved that in manifolds of dimension greater than or equal to 3, the Nielsen number is the minimal number of fixed points in the homotopy class. H Schirmer extended Wecken's result from fixed points to coincidences: in manifolds of dimension greater than or equal to 3, the Nielsen coincidence number is the minimal number of coincidences in the homotopy class.

Nielsen theory and Wecken theory raised many questions concerning the minimal number of fixed points of continuous functions in their homotopy classes, and finally in 1980's B Jiang solved many such questions (using Thurston theory on surfaces [12]). Upon Jiang's successful results, the fixed point theory again became the target of attention and until now, it is a research area in which many researchers are working actively and which is related to various topics such as Floer homology. ([3]).

3. Some Research Problems

The most useful tools among mathematical tools used in the fixed point theory is the (co)homology theories. The topological fixed point theory is not simply about the equation $f(x) = x$. For example, for two continuous functions f, g, we can consider the solutions of the equation $f(x) = g(x)$, i.e., the coincidences of f and g. Lefschetz's pioneering work is in the line of thinking the coincidence theory. For continuous functions $f, g : M \to N$ between closed orientable manifolds of equal dimension n, we have the following commutative diagram.

$$
\begin{array}{ccc}
H_q(M;\mathbb{Q}) & \xrightarrow{\;f_*\;} & H_q(N;\mathbb{Q}) \\[4pt]
\mu \big\uparrow \cong & & \nu \big\uparrow \cong \\[4pt]
H^{n-q}(M;\mathbb{Q}) & \xleftarrow{\;g^*\;} & H^{n-q}(N;\mathbb{Q}).
\end{array}
$$

Here the vertical homomorphisms are the Poincaré duality isomorphisms. Letting $\Theta_q = \mu g^* \nu^{-1} f_* : H_q(M;\mathbb{Q}) \to H_q(M;\mathbb{Q})$, the Lefschetz

coincidence number is defined as follows:

$$L(f, g) = \sum_{n=1}^{n} (-1)^q \text{trace}(\Theta_q).$$

If this homotopy invariant $L(f, g)$ is not 0, then there exists a point, a coincidence point, x such that $f(x) = g(x)$. Just like the Reidemeister trace has information both on the Lefschetz number and the Nielsen number in the study of the fixed point theory, we can deal with the Reidemeister trace in the study of the coincidence theory and it has information both on the Lefschetz coincidence number and the Nielsen coincidence number.

Another variation of the fixed point equation $f(x) = x$ is $f^n(x) = x$. A point x satisfying this equation is called a periodic point of f. The iterates $\{f^n\}$ of f forms a discrete dynamical system on X and the periodic points provide important information of the dynamical system. We define the following.

$$\text{Fix}(f) = \{x \in X \mid f(x) = x\},$$

$$P^n(f) = \text{Fix}(f^n),$$

$$P_n(f) = \text{Fix}(f^n) - \bigcup_{k<n} \text{Fix}(f^k),$$

$$\text{Per}(f) = \{n \in \mathbb{N} \mid P_n(f) \neq \emptyset\},$$

$$\text{HPer}(f) = \bigcap \{\text{Per}(g) \mid g \simeq f\}.$$

In this case, we define the minimal numbers of coincidence points up to homotopy as follows:

$$\text{NP}_n(f) \leq \min\{|P_n(g)| \mid g \simeq f\},$$

$$\text{NF}_n(f) \leq \min\{|P^n(g)| \mid g \simeq f\}.$$

An important problem is to find these homotopy invariants $\text{NP}_n(f)$, $\text{NF}_n(f)$ and another homotopy invariant $\text{HPer}(f)$, which is the set of homotopy minimal periods.

In order to deal with the size of fixed points in the topological fixed point theory, we need to assume that the spaces involved are compact. The spaces which are close to compact spaces are finitely dominated (FD) spaces. According to J H C Whitehead and J Milnor's conjecture, the FD spaces having torsion-free fundamental group are homotopic to compact spaces. When the space is $K(\pi, 1)$, this conjecture is the same as the following Serre conjecture: If π is of type FP, then it is of type FL. R Geoghegan developed the Reidemeister trace theory on the FD spaces X (which includes the Lefschetz theory and the Nielsen theory on the FD spaces). If X itself is a compact space, then the Reidemeister number of the identity function id_X is $R(\text{id}_X) = \chi(X)\langle 1 \rangle$, and so $L(\text{id}_K) = \chi(X)$. According as $\chi(X) = 0$ or $\chi(X) \neq 0$, we have $N(\text{id}_X) = 0$ or 1. When X is an FD space, it is conjectured that the identity function id_X has the Nielsen number 0 or 1. Related conjectures are projective class group conjecture, Whitehead torsion conjecture, D(2)-problem, and Bass conjecture. Consequently we can say that the Geoghegan's conjecture is the most important problem in the topological fixed point theory.

Many interesting and challenging problems are scattered in the Nielsen theory. But we will briefly introduce only 3 research topics based on the above discussion.

3.1. Fixed point theory and coincidence theory

One of important and difficult problems in the Nielsen theory is the computation of the Nielsen fixed point number and the Nielsen coincidence number. The Lefschetz number and the Reidemeister number are defined algebraically, however the Nielsen number is defined geometrically and so its computation is very difficult. But for any continuous function f defined on a torus, the Nielsen number is equal to the Lefschetz number up to sign. For the f, its lift to the universal covering space is homotopic to a linear map f_*, called a linearization of f. Then f satisfies $N(f) = |L(f)|$ (Anosov relation) and $L(f) = \det(I - f_*)$ (Computation Method) ([2]). A torus is a homogeneous space of the Abelian Lie group $\mathbb{R}^n$ by a lattice $\mathbb{Z}^n$ of it. Generalising this, we obtain homogeneous spaces such as nilmanifolds or solvmanifolds. Just like Kelin bottle, a Riemannian flat manifold which has a torus as a covering space, we can also consider infra-nilmanifolds and infra-solvmanifolds.

By the results of W Killing and H Hopf, the study of Riemannian flat manifolds is the same as that of the Euclid space forms problem. That is, an n-dimensional Riemannian flat manifold is isometric to the quotient space $\mathbb{R}^n/\pi$, where π is an n-dimensional Euclidean space form; π is a torsion-free discrete cocompact subgroup of $\text{Aff}(\mathbb{R}^n) := \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$. From L Bieberbach Theorem, π is a virtually $\mathbb{Z}^n$ group. Hence Riemannian flat manifold is a manifold having a torus as a covering space, an infra-torus manifold. The

Klein bottle is an example of such a manifold. L Auslander started the generalisation of the Euclidean space forms, and K B Lee and F Raymond completed its generalisation, called the nilpotent space forms. Due to M Gromov and E Ruh, the study of almost flat manifolds is the same as that of the nilpotent space forms problem. Hence the infra-nilmanifolds are almost flat manifolds. Algebraically defined (infra-)solvmanifolds were studied mainly by G D Mostow, H-C Wang, and L Auslander. But the Bieberbach's theorem which works for the Euclidean Lie groups and the nilpotent Lie groups is not true anymore for solvable Lie groups. Identifying manifolds corresponding to (infra-)solvmanifolds is the study in progress.

The computation of the Nielsen number on torus has raised questions, Problem of Anosov relation and Problem of computation method, of computing the Nielsen number on some classes of manifolds containing torus. Some known results until now are given below and also a related research problem is given.

- D V Anosov [1] (nilmanifolds):

$$N(f) = |L(f)| = |\det(I - f_*)|$$

- J Jezierski-C K McCord [16] (nilmanifolds):

$$N(f, g) = |L(f, g)|$$

- [14, 10] (Special solvmanifolds of type $(R) \supset$ nilmanifolds):

$$L(f, g) = \det(g_* - f_*).$$

It is known that the Anosov relation $N(f) = |L(f)|$ is not true on infra-homogeneous spaces.

- Averaging Formula for fixed points and co-incidences [11]:
  $f, g : M_2 \to M_2$, Ψ_i the holonomy group of infra-nilmanifolds M_i,

$$L(f, g) = \frac{1}{|\Psi_1|} \sum_{A \in \Psi_2} \det(g_* - A_* f_*),$$

$$N(f, g) = \frac{1}{|\Psi_1|} \sum_{A \in \Psi_2} |\det(g_* - A_* f_*)|.$$

Problem. Derive the averaging formula for the Nielsen number on (infra-)solvmanifolds.

3.2. Application to dynamical systems

One of the important examples in higher dimensional dynamics is an Anosov dynamical system. A linear transformation with integer coefficients induces a map on the standard torus and when this induced map is an Anosov automorphism, this map has a very rich dynamical structure. Namely, this map is a chaotic map. In particular, the set of periodic points of this map is a dense subset of the torus ([6]). The study of Anosov systems on (infra-)nilmanifolds has big progressed by A Manning ([15]). On the other hand, when the automorphism is replaced by an endomorphism, among other dynamical structures on infra-nilmanifolds, we have studied the density of the periodic points ([9]). That is, the eventually periodic points form a dense subset. Now we may raise the following question.

Problem. Every nil-endomorphism has dense periodic points.

On the other hand, S Smale defined the Lefschetz zeta function, which reflects periodic point structure of corresponding dynamical system, i.e., what kind of periods of periodic point classes for a discrete dynamical system f^n are possible ([18]).

$$L_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right).$$

In a similar way, we can define the Nielsen zeta function and the Reidemeister zeta function.

$$N_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right),$$

$$R_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n\right).$$

It can be shown easily that the Lefschetz zeta function is a rational function. A Felshtyn showed that the Nielsen zeta function and the Reidemeister zeta function are rational functions on nilmanifolds and then he raised the question that the Nielsen zeta function and the Reidemeister zeta function are rational functions on infra-nilmanifolds. Using the averaging formula (3.1), it is claimed in [5] that the Nielsen zeta function and the Reidemeister zeta function on infra-nilmanifolds are still rational. More generally,

Problem. ([7]). Determine on which spaces and for which functions the rationality and the algebraicity of the Nielsen zeta function and the Reidemeister zeta function.

In the topological discrete dynamical systems, the change of the minimal number of periodic points and the minimal period of periodic points

are observed in the homotopy category. Hence computing the Nielsen type numbers $NP_n(f)$ and $NF_n(f)$ which are lower bounds for the minimal number of periodic points is an important problem, and computing the minimal period $HPer(f)$ of periodic points is also an important problem ([13]). The problem on nilmanifolds was completed though, the problem on Riemannian flat manifolds except Klein bottle or some special cases of solvmanifolds are totally unknown. Recently some results have been obtained on 3-dimensional flat manifolds and the problem on infra-homogeneous spaces has been an interesting and active research topics for Nielsen theory people.

3.3. Bass conjecture

First of all we shall introduce briefly on Bass conjecture. For any group G, let C be the set of conjugacy classes $\langle g \rangle$ of G and consider the quotient (additive group-) homomorphism $q : \mathbb{Z}G \to \mathbb{Z}C$ from the group ring $\mathbb{Z}G$ of G to the free Abelian group $\mathbb{Z}C$. Let $A = (a_{ij})$ be a square matrix with entries in $\mathbb{Z}G$. Then the image of the trace of A under q is an element of $\mathbb{Z}C$. Denoting this element by $T(A)$, it can be written as a linear combination of elements of C, $T(A) = \sum n_g \langle g \rangle$ and we call $T(A)$ the Hattori–Stallings trace of A.

Problem. (Bass Conjecture). If $A^2 = I$ and $T(A) = \sum n_g \langle g \rangle$, then $n_g = 0$ for all $\langle g \rangle \neq \langle 1 \rangle$.

R Geoghegan developed the Nielsen theory on FD (finitely dominated) spaces. An FD space is a space from which there is a retraction onto a compact space under homotopy. Main idea was to lift the Nielsen theory from compact spaces to FD spaces. The Nielsen number of the identity on compact spaces is either 0 or 1.

Problem. (Geoghegan Conjecture). The Nielsen number of the identity on FD spaces is either 0 or 1.

When G is a finitely presented group, Geoghegan showed that his conjecture and Bass conjecture are equivalent. To understand this, we need to understand C T C Wall's finiteness obstruction

and so any detailed explanation is not appropriate here and the readers refer to the reference [8]. We close this article by emphasising again that this conjecture is the most important problem in the Nielsen theory.

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