Chern-Cheeger-Simons Theory

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1. Introduction

In the early twentieth century, in different branches of mathematics, the notion of local product structure, i.e., fiber spaces and their generalisations appeared. Characteristics classes are the simplest global invariants which measure the deviation of a local product structure from a product structure. They are closely related to "curvature" in differential geometry.

The simplest characteristic class is the Euler characteristic: given a finite cell complex M, the Euler characteristic is defined by

$$\chi(M) = \sum_k (-1)^k \alpha_k = \sum_k (-1)^k b_k.$$

Here α_k is the number of k-cells and b_k is the k-dimensional Betti number. If M is a compact oriented differentiable n-manifold, then $\chi(M)$ is also expressed as the number of zeroes of a smooth vector field on M, counted with multiplicities. This is generalised to looking at k generic smooth vector fields on M, and the loci where the exterior product of the vector fields vanish. This corresponds to a $(k-1)$-cycle with $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$-coefficients. The homology class is independent of the choice of the vector fields. This leads to Stiefel–Whitney cohomology classes $w_i \in H^i(M, \mathbb{Z}/2\mathbb{Z})$, for $1 \leq i \leq n-1$, $i = n-k+1$. When $k = 1$, the Euler class w_n has integer coefficients and is related to $\chi(M)$ as follows:

$$\chi(M) = \int_M w^n.$$

Whitney explored this notion further to consider vector bundles on arbitrary topological spaces. In particular he looked at universal vector bundles (more generally principal bundles) on complex Grassmannian manifolds $G(r, N)$. They parametrise r-dimensional subspaces W in a $(r+N)$-complex vector space. These manifolds have a simple structure, namely their odd dimensional cohomology groups are zero. The relevance of the universal bundle lies in the Whitney–Pontrjagin imbedding theorem, which says that a vector bundle of rank r on a finite cell complex M is induced by a continuous map $f : M \to G(r, N)$ for large N, and f is well-defined upto homotopy. Given a cohomology class α on $G(r, N)$, the pullback class $f^*\alpha$ is a characteristic class of the vector bundle on M. This gave rise to (universal) Pontrjagin classes and Chern classes. The universal Chern class c_i can be loosely described as the locus of r-subspaces W satisfying the relation:

$$\dim(W \cap V_{i+N-1}) \geq i,\ 1 \leq i \leq q.$$

Here N_{i+N-1} be a fixed vector subspace of $\mathbb{C}^{N+r}$ of dimension $i+N-1$. The pullback of c_i, via f, define the Chern classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$ for $1 \leq i \leq r$.

Other constructions of Chern classes are via Grothendieck's splitting principle and defining de Rham Chern classes using a smooth connection. The functoriality properties imply that all the constructions give the same class in cohomology with complex coefficients. These are the primary invariants of the vector bundle E.

2. Connections and Invariants

2.1. *Chern–Weil invariants*

A connection ∇ on a smooth complex vector bundle E, on a smooth manifold M is a structure which allows the differentiation of sections of E. It is a mapping

$$\nabla : \Gamma(E) \to \Gamma(T^* \otimes E).$$

Here T^* is the cotangent bundle on M and ∇ satisfies the following conditions:

$$\nabla(s_1 + s_2) = \nabla(s_1) + \nabla(s_2),$$

$$\nabla(f.s) = df \otimes s + f.\nabla(s)$$

for $s_1, s_2, s \in \Gamma(E)$ and f is a smooth function.

Given a local frame $\{s_1, s_2, ..., s_r\}$ for sections of E, we can write

$$\nabla(s_i) = \sum_j \theta_i^j \otimes s_j.$$

Here $\theta := (\theta_i^j)$ for $1 \leq i, j \leq r$ is a matrix of one-forms, called as the connection matrix. The effect of change of frame by A, a $r \times r$-nonsingular matrix whose entries are smooth functions, gives the connection matrix:

$$\theta' := dA.A^{-1} + A.\theta.A^{-1}.$$

Taking exterior differentiation of the connection matrix, we obtain

$$\Theta := d\theta - \theta \wedge \theta.$$

This is the "curvature" matrix of two forms. Under a change of frame, we have

$$\Theta' = A.\Theta.A^{-1}.$$

As a consequence, we note that the trace form $tr(\Theta^i)$ is a globally defined form of degree $2i$ on M. It is shown to be a closed form and corresponds to the de Rham Chern class $c_i^{dR}(E)$ in $H^{2i}(M, \mathbb{C})$. The cohomology classes are independent of the choice of connection. This is the Chern–Weil theory and their invariants.

2.2. *Flat connections and secondary invariants*

Assume that the connection is *flat*, in other words we have the vanishing of the curvature form Θ. This immediately implies that the de Rham Chern classes are identically zero, for each $i \geq 1$. However the vector bundle E need not be trivial on M. But it is close to being trivial, in the sense that Deligne and Sullivan proved that there is a finite covering $M' \to M$ such that the flat vector bundle pulled over M' is trivial as a smooth vector bundle. This geometrically captures the information that the integral Chern classes are torsion.

Now we look at the coefficient sequence:

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0.$$

This gives the exact sequence of cohomology groups:

$$\begin{array}{ccccccc} \to H^{2i-1}(M, \mathbb{C}/\mathbb{Z}) & \to & H^{2i}(M, \mathbb{Z}) & \to & H^{2i}(M, \mathbb{C}) & \to \\ & & c_i(E) & \mapsto & c_i^{dR}(E). & \end{array}$$

Since ∇ is flat we saw that $c_i^{dR}(E) = 0$. Hence the above exact sequence suggests that the integral Chern class has a lifting in $H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$. However, the lifting needs to be canonical and satisfy functorial properties. This prompted the search to define secondary invariants for flat connections on a complex vector bundle. Chern and Simons initiated this study and defined canonical secondary invariants denoted by $TP(\nabla)_i$ which lie on the total space of the vector bundle. In a later work by Cheeger and Simons, these invariants were defined on the base manifold, in an intermediate cohomology fitted in the above exact sequence, and which lift the integral Chern classes. This cohomology group is called the *differential cohomology* and the invariants are called the *differential characters*.

Recall the differential cohomology:

$$\begin{aligned} \widehat{H^p}(M) := \{(f, \alpha) : f : Z_{p-1}(M) \to \mathbb{C}/\mathbb{Z},\ \delta(f) \\ = \alpha, \alpha \text{ is a closed form and} \\ \text{integral valued}\}. \end{aligned}$$

Here $Z_{p-1}(M)$ is the group of $(p-1)$-dimensional cycles, and δ is the coboundary map on cochains. The linear functional $\delta(f)$ is defined by integrating the form α against p-cycles, and it takes integral values on integral cycles.

This cohomology fits in an exact sequence:

$$0 \to H^{p-1}(M, \mathbb{C}/\mathbb{Z}) \to \widehat{H^p}(M) \to A^p_{\mathbb{Z}}(M)_{cl} \to 0. \quad (1)$$

Here $A^p_{\mathbb{Z}}(M)_{cl}$ denotes the group of closed complex valued p-forms with integral periods.

Given a smooth connection (E, ∇) on M. The Chern forms $c_i(E, \nabla)$ are $2i$-differential forms on M.

The Chern form $c_i(\nabla) = tr(\Theta^i)$ in degree $2i$ corresponds to an element in the group $A^{2i}_{\mathbb{Z}}(M)_{cl}$. Now use the universal smooth connection (defined by Narasimhan–Ramanan), which lies on a Grassmannian $G(r, N)$, for large N. The differential cohomology of $G(r, N)$ is just the group $A^{2i}_{\mathbb{Z}}(M)_{cl}$, since the odd degree cohomologies of $G(r, N)$ are zero. Hence the differential character $\hat{c}_i$ is the Chern form of the universal connection. The pullback of this element via the classifying map of (E, ∇) defines the differential character $\hat{c}_i(E, \nabla) \in \widehat{H^{2i}}(M)$.

Suppose ∇ is flat, i.e., $\Theta = 0$. Then the Chern form $c_i(E, \nabla) = 0$. Hence the differential character lies in the group $H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$.

$$\hat{c}_i(E, \nabla) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Z}), \ i \geq 1.$$

These are the Chern–Cheeger–Simons invariants or the secondary invariant of flat connections.

2.3. *Closing remarks*

The above constructions are due to S S Chern, J Cheeger and J Simons. Their constructions and properties are discussed, and with applications in topology and geometry in the cited bibliography. Later, other constructions of the secondary invariants have also been given by A Beilinson, A Connes, M Karoubi and H Esnault. The Riemann–Hilbert correspondence associates to a flat connection its monodromy representation, i.e., a $GL_r(\mathbb{C})$-representation of the fundamental group of the base manifold. Cheeger–Simons proved that the secondary classes are rigid in degrees at least two, using a variational formula involving a one-parameter family of connections. Since the moduli space of $GL_r(\mathbb{C})$-representations has countably many connected components, this gave rise to the Chern–Cheeger–Simons question whether the secondary invariants are torsion in degrees at least two. A Reznikov gave an affirmative answer when the manifold is a smooth complex projective variety. The author and C Simpson prove this in a special case using logarithmic flat connections on a smooth complex quasi-projective variety with an irreducible smooth boundary divisor. These questions and theories also initiated an algebraic theory of flat connections on algebraic varieties by S Bloch and H Esnault with applications towards the theory of algebraic cycles.

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