Seifert Fibering

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A bundle E consists of a base space B and a fiber F. Over each point of the base space, there is a fiber attached. In general E is not just the product $B \times F$ but it is "twisted". E is called the *total space* and the map $p : E \to B$ is called projection. Therefore, $p^{-1}(b)$ is the fiber over $b \in B$. The most critical condition for a bundle is the local triviality. That is, for every $b \in B$, there exists a neighbourhood U such that $p^{-1}(U) \cong U \times F$, that is the product bundle. Now the *Seifert fibering* is a more general concept than fiber bundle in the sense that some fibers can be singular.

Seifert fibering was initiated by a paper in 1933 by Herbert Seifert. This is a theory naturally developed to study 3-dimensional manifolds, and it is still widely used to study low dimensional spaces. This theory precedes the fiber bundle theory.

As is well known [3, 4], there are 8 geometries in 3-dimension, and 6 of these have Seifert fiber structures. Of the remaining two, Sol-geometry has a generalised Seifert fiber structure. Therefore, the only 3-dimensional geometry that does not admit a Seifert fiber structure is the hyperbolic geometry. Classical 3-dimensional Seifert fibering has 2-dimensional base orbifold with 1-dimensional circle fiber.

A geometric structure consists of a model space X and a group acting on X. Usually the group of isometries $\mathrm{Isom}(X)$ is used. A *manifold M has a X-geometry* means that M has model space X. More precisely, there exists a discrete subgroup $\Pi \subset \mathrm{Isom}(X)$ which acts on X so that M is the quotient $\Pi\backslash X$. If the model space X has a genuine bundle structure $F \to X \to B$, and if Π preserves the fibers, then $\Pi\backslash X$ may not be a genuine fiber bundle, but may be a fiber bundle with singularity.

All the model spaces for classical Seifert fiberings have fiber bundle structure $F \to X \to W$, where $F = \mathbb{R}^1$ or S^1 (circle), and $W = \mathbb{R}^2$, S^2 or $\mathbb{H}^2$. Therefore there are 6 combinations.

$$\begin{aligned}
&\mathbb{R} \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^2,\\
&\mathbb{R} \longrightarrow \mathbb{R} \times S^2 \longrightarrow S^2,\\
&\mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{H}^2 \longrightarrow \mathbb{H}^2,\\
&\mathbb{R} \longrightarrow \mathrm{Nil} \longrightarrow \mathbb{R},\\
&S^1 \longrightarrow S^3 \longrightarrow S^2,\\
&\mathbb{R} \longrightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})} \longrightarrow \mathbb{H}^2.
\end{aligned}$$

The model spaces for general Seifert fibering are principal bundle $G \to P \to W$, where G is a Lie group, and W is a manifold. A *Principal bundle* means that G acts on P freely, and properly. We require G to have lattices. Also we consider only the fiber-preserving homeomorphisms. Namely,

$$\mathrm{Top}_G(P) = \{\text{weakly } G\text{-equivariant homeomorphisms of } P\}.$$

The six 3-dimensional Seifert fibering geometries X have $\mathrm{Isom}(X)$ which are fiber-preserving. (In the case of $\mathbb{R}^3$, one needs to choose a fiber in advance). In other words, except for $\mathbb{R}^3$,

$$\mathrm{Isom}(X) \subset \mathrm{Top}_G(X),$$

where $G = \mathbb{R}^1$ or S^1.

In the case of $\mathbb{R}^3$, it is not true that $\mathrm{Isom}(\mathbb{R}^3) \subset \mathrm{Top}_{\mathbb{R}^1}(\mathbb{R}^1 \times \mathbb{R}^2)$, but Π can be embedded in $\mathrm{Isom}(\mathbb{R}^3) \bigcap \mathrm{Top}_{\mathbb{R}^1}(\mathbb{R}^1 \times \mathbb{R}^2)$. Lastly, $\mathrm{Sol} = \mathbb{R}^2 \rtimes \mathbb{R}$ is a Seifert fibering whose fiber is 2-dimensional $\mathbb{R}^2$ and base is $\mathbb{R}$.

Let G be a connected Lie group, W a manifold and $G \to P \to W$ a principal fibration. As in the case of 3-dimension, if a discrete group of $\Pi \subset \mathrm{Top}_G(P)$ acts on P freely, then we obtain a space $M = \Pi\backslash P$. If $\Pi \cap G = \Gamma$ is a lattice of G, the fibering of the model space yields

$$\Gamma\backslash G \longrightarrow M \longrightarrow W/Q,$$

where $Q = \Pi/\Gamma$. In general, M is not a genuine fibering. Moreover, W/Q is not a manifold but an orbifold. The homogeneous space $\Gamma\backslash G$ is called the *typical fiber*, and *singular fibers* are finitely covered by $\Gamma\backslash G$. The groups G used in Seifert fibering are $\mathbb{R}^n$, nilpotent, or some special solvable Lie groups and semi-simple Lie groups. G always must have a lattice Γ, and singular fibers are finite quotients of $\Gamma\backslash G$. These singular fibers are called *infra-homogeneous spaces*.

We need to understand these fibers. Consider the case when $G = \mathbb{R}^n$ (a commutative Lie group). The following are asked by Hilbert in 1900, and proved by Bieberbach in 1909. These are known as Bieberbach theorems. As is well known, the group of isometries of $\mathbb{R}^n$ is $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n)$, which is a subgroup of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$. A discrete cocompact subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ is called a *crystallographic group* (=CG).

(**B1**) If $\Pi \subset \mathrm{Isom}(\mathbb{R}^n)$ is a CG, then $\Pi \cap \mathbb{R}^n = \mathbb{Z}^n$, and is a lattice of $\mathbb{R}^n$. Therefore, $\Pi/\mathbb{Z}^n = \Phi$ is finite.
(**B2**) If two CG $\Pi, \Pi' \subset \mathrm{Isom}(\mathbb{R}^n)$ are isomorphic, the isomorphism is conjugation by an element of $\mathrm{Aff}(\mathbb{R}^n)$.
(**B3**) In every dimension $n > 0$, there are only finitely many crystallographic groups up to isomorphism.

In general, an action of a crystallographic group Π on $\mathbb{R}^n$ is not free. Therefore, $\Pi\backslash\mathbb{R}^n$ is an orbifold. For this action to be free, it is necessary and sufficient that the only element of Π which has finite order is the identity. When this happens, $\Pi\backslash\mathbb{R}^n$ is a manifold. Since Π acts on $\mathbb{R}^n$ as a covering transformation (properly discontinuously and freely), $M = \Pi\backslash\mathbb{R}^n$ is locally isometric to $\mathbb{R}^n$. Therefore, the sectional curvature is 0 at every point, and to every direction, hence M is a flat manifold. Conversely, if M is an n-dimensional flat manifold, then it should be of the form $M = \Pi\backslash\mathbb{R}^n$, where Π is an n-dimensional torsion free crystallographic group.

The 3 statements above hold for G nilpotent (more generally, some solvable) Lie groups. The simplest example of nilpotent Lie group is the 3-dimensional Heisenberg group

$$\left\{ \begin{bmatrix} 1 & x & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, x \in \mathbb{R} \right\}.$$

This is called Nil, and is one of the eight 3-dimensional geometries.

Let G be nilpotent. A discrete cocompact subgroup Π of $\mathrm{Isom}(G)$ is called an *almost crystallographic group* (=ACG). As in the case of $G = \mathbb{R}^n$, it is known that $\Pi \cap G$ is a lattice of G. In fact, it is proved in 1960–1982 that all the statements (**B1**)–(**B3**) for commutative G can be generalised to nilpotent groups G [2, Chapter 8].

(**B1**)–(**B3**) have the following topological consequences. When $\Pi \subset \mathrm{Isom}(G)$ is ACG, a necessary and sufficient condition for the action of Π on G is free is that Π is torsion free. According to the first statement, $\Gamma = \Pi \cap G$ is a lattice of G, and the quotient $\Pi/\Gamma = \Phi$, called the *holonomy group*, is a finite group. Homogeneous space $\Gamma\backslash G$ is a *nilmanifold*, and its finite quotient $\Pi\backslash G = (\Gamma\backslash G)/\Phi$ is an *infra-nilmanifold*.

Almost flat manifolds introduced by Gromov [1] requires that the sectional curvature lies near 0. More precisely, M is *almost flat* if, for every $\epsilon > 0$, there exists a Riemannian metric g_ϵ such that $\mathrm{diam}(M, g_\epsilon) \leq 1$ and the sectional curvature satisfies $|K_{g_\epsilon}| < \epsilon$. Theorems of Gromov–Ruh state that M is almost flat if and only if M is an infra-nilmanifold. That is,

$$\{\text{almost flat manifolds}\} = \{\text{infra-nilmanifolds}\}.$$

Generalised Bieberbach theorems (topological version) are as follows.

(**B1**) Every infra-nilmanifold is finitely covered by a nilmanifold.
(**B2**) Homotopy equivalent infra-nilmanifolds are affinely diffeomorphic.
(**B3**) Under a nilmanifold, there are only finitely many infra-nilmanifolds which are essentially covered by the nilmanifold. Here essential covering means that all the covering transformation except the identity is homotopically non-trivial.

For example, for $G = \mathbb{R}^n$, $\Gamma\backslash G = T^n$ (torus), and all singular fibers are flat manifold (sectional curvature = 0), finitely covered by the flat torus. There are many application of this *generalised Seifert fibering*.

1. Given a group Π, does there exist a $K(\Pi, 1)$-manifold? Of course, Π must be torsion free, and satisfy Poincaré duality, etc. However, a complete condition for Π to have a $K(\Pi, 1)$-manifold is not known. Unless we give strong conditions to Π, it seems impossible to answer this question. However, for many interesting groups Π, Seifert fibering space construction shows that such $K(\Pi, 1)$'s exist. In order to work with Π, we need some condition. We assume that Π contains Γ, a lattice of a Lie group G, as a normal subgroup. Thus we have a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow Q \longrightarrow 1.$$

Now assume Q acts on a space W. Using the actions (Γ, G) and (Q, W), we try to find an action of Π on $G \times W$. More generally, we can think of, not just product $G \times W$, but even a principal bundle $G \to P \to W$. In case G is nilpotent or some special solvable Lie group, Seifert fiber construction yields an action of Π on P, and a manifold $\Pi \backslash P$.

2. Two homotopy equivalent spaces cannot be the same in general. Poincaré conjecture which was solved positively some time ago is such an example: "If a manifold M is homotopy equivalent to a 3-dimensional sphere S^3, then it is homeomorphic to S^3". There is strong rigidity among Seifert fiber spaces. In many cases, it is known that, among Seifert fibers, same homotopy implies homeomorphism. Moreover, such a homeomorphism can be found to preserve the fibers. For example, the rigidity for flat manifolds is: If a closed manifold has same homotopy as a flat manifold, then they are homeomorphic.

3. Let M be a $K(\Pi, 1)$-manifold, F a finite subgroup of $\mathrm{Out}(\Pi)$. $\mathrm{Out}(\Pi) = \mathrm{Aut}(\Pi)/\mathrm{Inn}(\Pi)$ is equal to $\pi_0(\mathcal{E}(M))$, and it can be thought as the group of homotopy classes of self-homotopy equivalences. For a 2-dimensional surface M, it was an outstanding question whether such a group F can be realised as a group acting on M, this was known as the Nielsen realisation problem, which was solved in 1980. In general, for $K(\Pi, 1)$-manifolds which are Seifert fiber spaces, the same question arises. For many Lie groups, Seifert fiber construction yields positive conclusion for the realisation problems.

If F acts on M, we can form the extended group $\widetilde{F}$ so that

$$1 \to \Pi \to \widetilde{F} \to F \to 1$$

is exact, and its action on $\widetilde{M}$. Our question is, conversely, F and $\varphi : F \to \mathrm{Out}(\Pi)$ are given. Such a homomorphism φ is called an *abstract kernel*. Given φ, the obstruction for the existence of such short exact sequence lies in $H^3(F; \mathrm{Centre}(\Pi))$. It is often difficult to prove such an extension $\widetilde{F}$ exists. Suppose there is an extension. If Π is the fundamental group of a Seifert fibering, a short exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$ already exists, then we can do Seifert fiber construction with

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{F} \longrightarrow \widetilde{F}/\Gamma \longrightarrow 1.$$

Here we need to check if Γ is a normal subgroup of $\widetilde{F}$. If we note that $\widetilde{F}/\Gamma$ is a finite extension of Q, this construction is not too difficult.

4. If a manifold can be written as a direct product, it will be easy to study the object because lower dimensional spaces are easier. Suppose a torus T^k acts on a manifold M. Then

$$\mathrm{ev} : (T^k, e) \longrightarrow (M, x), \quad \mathrm{ev} : t \longmapsto t \cdot x$$

is called an evaluation map. If M is a $K(\Pi, 1)$-manifold,

$$\mathrm{ev}_\# : \pi_1(T^k, e) \longrightarrow \pi_1(M, x)$$

is known to be injective. If

$$\mathrm{ev}_* : H_1(T^k; \mathbb{Z}^k) \longrightarrow H_1(M; \mathbb{Z}^k)$$

is injective, this T^k-action is said to be *homologically injective*.

Theorem. Suppose T^k acts on M. The following are equivalent.

1) The map

$$\mathrm{ev}_* : H_1(T^k; \mathbb{Z}) \to H_1(M; \mathbb{Z})$$

induced by $\mathrm{ev} : (T^k, e) \to (M, x)$ is injective.

2) $\pi_1(M)$ virtually split. That is, there exists a finite indexed normal subgroup $\mathbb{Z}^k \times Q$ of $\pi_1(M)$, where $\Phi = \pi_1(M)/\mathbb{Z}^k \times Q$ is finite commutative.

3) M almost splits. That is,

$$M = (T^k \times N)/\Phi,$$

where N is $(\dim(M) - k)$-dimensional manifold, and Φ is a finite commutative group acting on $T^k \times N$ diagonally; as translations on the T^k-factor.

Homologically injective (i.e., ev_* is injective) is much stronger than injective (i.e., $\mathrm{ev}_\#$ is injective); is much rarer. For example, in the Nil-geometry, Centre(Nil) induces S^1-action, which is never homologically injective, because the lattice Γ of Nil is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, $1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 1$ (exact), but Γ is not the product $\mathbb{Z} \times \mathbb{Z}^2$.

5. Many Seifert fibered spaces are known to have maximal torus action, and this helps to split the spaces. This is based on the following facts. If G is a connected compact Lie group acting on a $K(\Pi, 1)$-manifold, G must be torus. Furthermore, $\mathrm{ev}_\# : \pi_1(T^k, e) \to \pi_1(M, x)$ is injective, and $\mathrm{ev}_\#(\pi_1(T^k, e))$ lies in the centre of $\pi_1(M, x)$. Therefore, if the centre of $\pi_1(M, x)$ is $\mathbb{Z}^k$, there is an open question whether there exists a T^k-action on M. In fact, when M is a compact manifold, it

is still unknown if the centre of $\pi_1(M)$ is finitely generated. Seifert fiber technique yields many results on this question.

6. The second Bieberbach theorem can be generalised even more, and it is a good tool in fixed point theory. Let G be a simply connected Lie group, Endo(G) the set of all endomorphisms of G. This is not a group under composition (since there is no inverse elements), but form a semi-group. For example, if $G = \mathbb{R}^n$,

$$\mathrm{Endo}(\mathbb{R}^n) = \mathfrak{gl}(n, \mathbb{R}) \text{ (= all } n \times n \text{ matrices).}$$

Therefore, $\mathrm{aff}(G) := G \rtimes \mathrm{Endo}(G)$ is also a semi-group, and acts on G:

$$(a, A)(b, B) = (a \cdot Ab, AB)$$
$$(a, A) \cdot x = a \cdot Ax.$$

Theorem. Let $\Pi, \Pi' \subset \mathrm{Aff}(G)$ be two ACGs. For any homomorphism $\theta : \Pi \to \Pi'$, there exists $g = (d, D) \in \mathrm{aff}(G)$ such that

$$\theta(\alpha) \circ g = g \circ \alpha$$

for all $\alpha \in \Pi$.

If θ is an isomorphism, then g is invertible, and $\theta(\alpha) = g \circ \alpha \circ g^{-1}$ (i.e., conjugation by g). (See the Bieberbach theorem (**B2**)). $\theta(\alpha) \circ g = g \circ \alpha$ is called semi-conjugate. The power of this theorem is, when calculating homotopy invariants, one can replace any continuous map by an affine map. Since affine map behaves so nicely, many calculations become easy. Let $M = \Pi \backslash G$ be an infra-nilmanifold and $f : M \to M$ a continuous map. Then the Lefschetz number $L(f)$ and Nielsen number $N(f)$ are such examples. The map f induces an endomorphism θ of $\pi_1(M) = \Pi$, and by the theorem above, it is semi-conjugate to an affine map (d, D). In general, the lattice $\Pi \cap G$ of G may not be θ-invariant, there exists a θ-invariant sublattice Λ, and $\theta : \Lambda \to \Lambda$ can be found as a unique extension, as an endomorphism $E : G \to G$, $g \mapsto dD(g)d^{-1}$. The most important formula is the following averaging formula.

Theorem. Let $M = \Pi \backslash G$ be an infra-nilmanifold with holonomy group Φ, $f : M \to M$ a continuous map. Then

$$L(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \det(I - A_* f_*),$$

$$N(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(I - A_* f_*)|,$$

where f_* is the linearisation of f, $E_* : \mathfrak{g} \to \mathfrak{g}$.

There are other applications of Seifert fiberings in the study of manifolds. Main tools are group cohomology, differential geometry, and Lie theory. The most recent developments can be found in the references of the book *Seifert fibering* [2].

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