On Some Non-Hermitian Ensembles of Random Gaussian Matrices*

Anthony Mays

In the broadest sense, a random matrix is merely a matrix with random numbers in it. In practice these matrices often have some symmetry or unitarity conditions, and the entries are frequently drawn from a Gaussian distribution. We call the set of matrices having such-and-such definition an ensemble. The main focus of the field of random matrix theory is to analyse the eigenvalue distribution of the ensemble, although the behaviour of the eigenvectors may also be of interest. It may be expected a priori that the eigenvalues of a random matrix are scattered uniformly at random over their support (exhibiting the “clumpy” patterns typical of such data), however this is far from true and they instead display strongly correlated behaviour. The most striking instance of this arises from the ubiquitous appearance of the product $\prod_{j<k} |\lambda_k - \lambda_j|^{\beta}$ in the eigenvalue joint probability density function (jpdf), which is the required Jacobian when changing variables from the matrix entries to the matrix eigenvalues. (We will discuss the $\beta$ parameter below.) Since this product implies that there is a low probability of finding two eigenvalues close to each other, the product is interpreted as eigenvalue repulsion. This repulsion is found in both Hermitian and non-Hermitian matrix ensembles. Another effect of the Vandermonde product is that, by Vandermonde’s well-known identity, the eigenvalue jpdf can be rewritten as a determinant. This latter fact is key to the analysis, allowing the powerful tools of linear algebra to be brought to bear on the problem.

Random Hermitian Gaussian matrices found some early applications in physics, largely due to the work of Eugene Wigner. The problem being faced at the time was the analysis of the highly excited states of heavy nuclei. Modelling the problem as a set of interacting particles quickly leads to a set of unwieldy coupled equations. Rather, Wigner [10] suggested that a statistical approach might be more useful, and he conjectured that the distribution of the spacing between energy levels will be well approximated by the eigenvalue spacing distribution of large, real symmetric, Gaussian matrices (this statement became known as the Wigner surmise); this expectation was also proposed by Landau and Smorodinsky [7, Lecture 7]. A few years later, it was comprehensively demonstrated that the repulsive nature of the energy levels could indeed be modelled by eigenvalues of symmetric matrices, and that the results matched Wigner’s predictions. Building on this work, Dyson [2] found that these random matrices naturally fell into three classes: real symmetric, complex Hermitian and real quaternion self-dual; a result which is fundamentally due to a theorem of Frobenius that there are exactly three associative division algebras over the reals. It turns out that the parameter $\beta$ in the Vandermonde product above corresponds to the number of independent real components in these matrix entries: $\beta = 1$ for real ensembles, $\beta = 2$ for complexes and $\beta = 4$ for real quaternions. These values for $\beta$ have become known as the Dyson indices. The ensembles corresponding to these values of $\beta$ are called the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles respectively, due to the invariance properties of their density functions.

Analogous ensembles (which have come to be known as Ginibre ensembles) of non-symmetric real, non-Hermitian complex and non-self-dual real quaternion matrices (with Gaussian entries) were introduced in [4], with matching Dyson indices. However, the lack of unitary diagonalisability caused significant complications for the real case, meaning that analytical progress was delayed until the early 1990s. For these non-Hermitian matrices, it is now standard practice to sacrifice orthogonal/unitary diagonalisability for

---

*This article is based on the results of my research made possible by the AustMS Lift-Off Fellowship, which allowed me to travel to Germany, attending a conference in Bonn and visiting the universities of Bielefeld and Regensburg. A tangible result of this award is the pre-print [8], “A real quaternion spherical ensemble of random matrices”.

---

January 2014, Volume 4 No 1
orthogonal/unitary *triangulisability* by using the Schur decomposition. Whereas the eigenvalues of Hermitian matrices are all real, the eigenvalues of non-Hermitian matrices are (in general) complex: $N \times N$ complex matrices have $N$ independent complex eigenvalues; $2N \times 2N$ real quaternion matrices have $N$ complex-conjugate pairs of eigenvalues; and $N \times N$ real matrices have $k$ real eigenvalues and $(N - k)/2$ complex-conjugate pairs. Further, given the eigenvalue repulsion implied by the Vandermonde product, we note that the complex-conjugate pairs in the real and real quaternion ensembles result in an effective repulsion of eigenvalues from the real axis (except for the strictly real eigenvalues in the real ensembles). This is demonstrated in the simulations of Fig. 1. Note that (except for near the real line) the eigenvalues are roughly uniformly distributed on a disk, which, before scaling is of radius proportional to $\sqrt{N}$. This is a consequence of a universality result — called the *circular law* — which says that in the limit of large matrix dimension the eigenvalue density, suitably scaled, of any matrix with independently and identically distributed (iid) entries (assuming finite mean and unit variance) is uniform on the unit disk [9]. The analogous result for Hermitian ensembles is the *semi-circular law*: that the (scaled) eigenvalue density is a semi-circle of unit radius [1].

If we now take two matrices $A, B$ from the same Ginibre ensemble and form the product $Y = A^{-1}B$, then we have formed a so-called *spherical ensemble*. To see the origin of the name, we stereographically project the eigenvalues of simulations of real, complex and real quaternion spherical ensembles in Fig. 2. The complex ensemble yields eigenvalues that are uniformly distributed over the sphere; the real ensemble has a clear ring of eigenvalues on a great circle corresponding to the real line; and the eigenvalue density for the real quaternion ensemble splits into two disjoint hemispheres along the same great circle corresponding to the real line. We say that these matrices are in the spherical universality class because of the spherical law, which is analogous to the circular and semi-circular laws discussed above. It states that for two matrices with iid entries, having mean zero and finite variance (in other words, obeying the circular law), then in the limit of large matrix dimension the eigenvalues of the product $Y = A^{-1}B$ are uniformly distributed on the sphere (after stereographic projection).

The eigenvalue correlation functions for the complex ($\beta = 2$) case were studied in [6] and the real ($\beta = 1$) case in [3]. The work done during
this Fellowship, and contained in [8], was on the
calculation of the correlation functions for the en-
semble corresponding to the last of the Dyson in-
dices, real quaternion ($\beta = 4$). The procedure first
involves changing variables from the elements of
$A$ and $B$ (each an $N \times N$ matrix with Gaussian
real quaternion entries) to the elements of
$Y = A^{-1}B$, which entails integrating out $4N^2$
degrees of freedom; then further changing variables to the
eigenvalues of $Y$, integrating out another $4N(N-1)$
variables, leaving just the $2N$ complex eigenval-
ues. Next, a Pfaffian\(^b\) form of the ensemble av-
erage is written down and we find polynomials
(called skew-orthogonal polynomials) that will skew-
diagonalise the matrix in the Pfaffian. Armed with
these polynomials we are then able to obtain the
eigenvalue correlation functions by functional
differentiation of the ensemble average. Lastly, we
take various limits to find that the eigenvalues are
indeed uniformly distributed over the sphere for
large $N$ (a consequence of the spherical law), and
further, that by zooming in on the sphere near
the great circle corresponding to the real line, we
recover the same correlation functions as in the
real Ginibre ensemble [5].

References

of Large Dimensional Random Matrices, Mathematics
Monograph Series, 2 (Science Press, Beijing, 2006).

of symmetry groups and ensembles in quantum
mechanics, *Journal of Mathematical Physics* 3 (1962)
1200–1215.

for the Gaussian real generalised eigenvalue prob-
lem, *Probability Theory and Related Fields* (10 April
2011), 1–47.

[4] J. Ginibre, Statistical ensembles of complex, quater-
nion and real matrices, *Journal of Mathematical
Physics* 3 (1965) 440–449.

[5] E. Kanzieper, Eigenvalue correlations in non-
Hermitian symplectic random matrices, *Journal of

[6] M. Krishnapur, Zeros of Random Analytic Func-
tions, PhD thesis, University of California, Berkeley


[8] A. Mays, A real quaternion spherical ensemble
of random matrices (2012). Available at

[9] T. Tao, V. Vu and M. Krishnapur, Random matrices:
Universality of ESDs and the circular law, *Annals of

[10] E. P. Wigner, On the statistical distribution of the
widths and spacings of nuclear resonance levels,
*Proceedings of the Cambridge Philosophical Society* 47
(1951) 790–798.

\(^b\)For an even dimensional matrix $X$, which is skew-Hermitian
($X^\dagger = -X$), $\text{Pf}(X) = \sqrt{\det X}$. 

This article was first published in the Gazette of the Australian
Mathematical Society and is republished with the kind permission
of the Australian Mathematical Society which retains the copyright.

**Anthony Mays**

Alcatel-Lucent Chair on Flexible Radio, Sup’elec, 3 rue Joliot-Curie, Plateau de Moulon, 91192 Gif-sur-Yvette, France

anthony.mays@supelec.fr

After growing up in the suburbs of Melbourne, Anthony Mays completed a Bachelor degree in economics and Mandarin Chinese at Monash University. He then under-
took a second degree in mathematics and physics before moving to Melbourne
University to graduate with honours in mathematics. Continuing on at Melbourne
University with a PhD under Peter Forrester, he studied the eigenvalue correlation
functions of various random matrix ensembles. The title of his doctoral thesis was
A geometrical triumvirate of real random matrices, which he submitted in 2011.
Since then, he has been a research assistant at Vanderbilt University, and has been
awarded a LiftOff Fellowship by the Australian Mathematical Society. He is currently
a postdoctoral researcher at Supélec in France. Anthony is also keen juggler and
has a strong interest in the mathematics of juggling — the title of his Honours thesis
was Combinatorial aspects of juggling, for which he received the Wyselaskie
Scholarship in Mathematics.