On Philosophy of Science and Noether's *Chefd'oeuvre*

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Abstract. Emmy Noether, being a mathematician, proved an exceptional result in Physics, which goes by the name of Noether's theorem. Starting with the philosophy of science, I try to discuss the theorem and its background. Invoking a pragmatic view of the same, I conclude it from a philosophical footing, which unveils its inner beauty. Though the philosophical implications of Noether's theorem have been studied, this article offers more refined version of it.

Keywords: Noether's theorem; symmetry; principle of least action; Lagrangian; Hamiltonian; ontology and epistemology.

1. Introduction

Though Omar Khayyám in his *magnum opus* Rubáiyát [1], expresses the limitation of humanistic inquiry as

> Then to the rolling Heaven itself I cried,
> Asking, "What Lamp had Destiny to guide
> Her little Children stumbling in the Dark?"
> And *"A blind understanding!"* Heaven replied.

However, rationality ingrained in human brain, took it with a pinch of salt. Going further in reducing the multifaceted edifice of reality, axiomatically, humans tried to disseminate the palatable chunks of acquired information through a language, whose semantics/syntax are constructed upon inter-subjective agreements. Questions that pertain to the very existence of objective reality are abstruse, and hence are pushed under the carpet of philosophy. Discussion about reality (*satyá*[a]), for instance in Eastern philosophy, dates back at-least to Vedic age. The very task of reaching out to objective reality, with our senses (which are subjective), turns out to be futile. In Western Philosophy, Rene Descartes presented the same argument in a concrete way, which goes under the name of Cartesian Skepticism [4], where he concludes the impossibility of forming an objective basis of reality, arguing how our sense perceptions qua experiences of the outer world, can be cheated by a "demon". Later on Plato through allegory of cave (Republic [5]), metaphorised the same, comparing ourselves to prisoners in a cave, whose inferences are based upon the conditioned (subjective) knowledge gained from the past. Such prisoners, who never stepped out of the cave, are in a delusion, inferring that the objects in the external world look exactly like their shadows formed on the wall of the prison, which *de facto* can be distorted versions of reality (out-there). One can go further generalising it, as if we are present in a set of infinitely embedded caves, and we may be seeing the reality in an infinitely distorted way. There is no way of knowing/authentically verifying it, unless we reach out to it. On the other hand one can conclude that the objective truths, formulated so far, strictly speaking are subjectively objective. We all agree upon a collective common set of axioms, and subjected to certain sense perceptions and inferences we arrive at a particular consistency (truth) which to strictly speaking is subjective, but can be relatively objective. This notion is essential to further our understanding.

Epistemology deals with the nature of knowledge obtained, from our cognitive and sense percepts (how the system appears to us), where as ontology is the study of the system as it is, independent of our perceptions, inferences, heuristics and empiricism. Epistemological categorisation adhering to a phenomenological basis, thereafter engendered various disciplines within science, mathematics being a cornerstone hitherto. Applying a set of tools of one discipline, to another discipline, gave rise to many more fields, for instance quantum biology or mathematical physics, etc., Praxis of physics adheres to experimental philosophy. Empirical data (outcomes of Spatiotemporal measurements) having an epistemological basis, is used to fabricate mathematical formalism, thereby obtaining a formula, which can predict outcomes of the experiment.

[a] As per Hermann Graßmann's *Wörterbuch zum Rig-Veda* [18] means "Speaking about that, which truly coincides with the real fact, and which is unchangeable" (*Wahr von der rede, die mit der wirklichen tatsache übereinstimmt....*).

Experimental predictions account to beliefs, where as experimental outcomes represent (observable) reality. For a theory to be successful, it should on one hand maintain harmony between belief systems and reality, and on the other hand should be consistent[b] with well established inferences. Updations in belief systems are done (by adding fudge factors into mathematical formula), to account for counterintuitive experimental results. Starting with an epistemological basis (context dependent and empirical) upon which a theory is constructed, it is hard to think about the extent to which it entails the underlying ontology of the system, as ontology surpasses empiricism, and it concerns with the intrinsic properties of the system. Though epistemologically obtained observable properties about an object give us some information about it, we cannot for sure infer that those properties are innate properties of the object itself. For instance, Redness of an apple, may not be the real ontological (objective) property of apple, but is just the epistemic state, corresponding to our sense of vision, which gets excited by the light that is reflected from apple. So redness of apple is an epiphenomenon of apple interacting with vision field. So in realism, by assuming that, what we see is really *out there*, one can keep such questions on hold. Many physicists are realists. To sum up, reality out there is perceptionally rich, semantically neutral, until epistemically viewed, and underlying ontological implications are subjected to skepticism. So Noether's theorem plays an important role in connecting the epistemic and ontic descriptions, and we will see it soon. At this point I would like to say a few words about Noether.

Emmy Noether was a German mathematician, who is best known for her groundbreaking contributions in the fields of mathematics and theoretical physics. Despite all the difficulties she encountered because of her gender, she has proved some of the most beautiful results which will stand forever. People in those days, subscribed to this very slogan, of describing the duties of a woman, Kinder, Küche, Kirche (Children, Kitchen, Church), and Noether endured years of poor treatment, both as a student and in her career. She was not allowed to teach under her own name, but few other seminal contemporary mathematicians quickly realised her talent and supported her. Hilbert being one among them, supported her, and sarcastically said, that university is not a bathhouse to show gender discrimination and Einstein once expressed her greatness, saying that Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. I will stop here, expressing my inability to describe her gigantic stature in this small article, by saying, "*Hanc marginis exiguitas non caperet*" and point to some of the interesting biographical notes on her [15, 16]. Now I will come to the actual topic, of Noether's theorem. As Jacobi who worked on Elliptical integrals, all his life, formulated canonical transformations which play a crucial part both in classical and quantum mechanics, Noether who worked all her life on abstract algebra formulated this beautiful theorem.

2. Noether's First Theorem

Theorem 2.1. *For every continuous symmetry (of Lagrangian, to say) there is a corresponding conservation law.*

Before going into the theorem, we need to understand the words, symmetry, invariance, conservation and Lagrangian. As we have discussed physics is constructed upon the framework of mathematical formalisms. The beauty of mathematics lies in its abstractness. Representation serves the role of mid-wife in bringing out aesthetics. For instance, even if you write this very statement $2 > 3$, on a wall, with the number 2 written in a bigger font than number 3, you still can infer that it is wrong, twoness of 2, is never greater than threeness of 3. Twoness and threeness are not an innate properties of the corresponding representational elements 2 and 3 respectively, but they lie in platonic world,[c] which has to be seen through the eye of mind. Consistency being an important consideration within science, demands epistemological constructs (taking futuristic up-

[b]Consistent with the definition of truth, which is axiomatically defined. Aristotle mentions in his Metaphysics V, "To say of what is, that it is, or of what is not, that it is not is true".

[c]Number systems are mere representations of platonic elements, so are the geometric shapes, that we are familiar within our day to day life. It is not far to see that, the line we draw on board, strictly will not satisfy Euclidean postulate about line, which as per that should have strictly one dimension, where as the line we see has a negligible (but non-zero) breadth, and so is a point, and so are all shapes that we study. They should be understood as representations of platonic shapes which are ideal.

dations into account) to be paradox-free. At this point, I should mention that, David Hilbert, proponent of formalism, tried to establish consistency in any axiomatic system, by the notion of metamathematics. Later on Kurt Gödel, through his celebrated incompleteness theorems went further imposing fundamental limitations on any axiomatic system. He showed that consistency can be achieved, but at the cost of missing out on the completeness. Pure mathematics operating in platonic realm, demands internal consistency, but worries less about the ontological implications.

Unlike physicists, the framework of mathematics works on maintaining one to one correspondence, between platonic realm, and corresponding representation set. Nevertheless, few mathematicians, go applying those platonic thoughts and formalisms, to real world phenomenon. Sometimes, there are very interesting connections between abstract theories and real world phenomenon, so interesting that even founders of such theories, never imagined the impact of their theories, in real world. Symmetry is one such thing that has got very interesting applications. Symmetry refers to the "sameness" or "similarity" of an object. Study of symmetry helped a lot, in understanding the hidden laws of universe. Symmetry entails invariance. A square has something called four-fold symmetry. If you rotate the square through an angle of 90°, with respect to the horizontal axis, it appears to be the same, entailing the fact that shape of the square is invariant under any such rotations. Sphere is infinitely symmetrical, as it looks the same, in which ever way we rotate it. So symmetry can be termed as immunity (thereby making it invariant) of the shape to look the same when it is subjected to certain changes (transformations). This very property of invariance plays an important role within mathematics. There are numerous invariants discovered, till date, which have a contextual existence. Dynamism being a central concept, in sciences, one likes to study such invariants, as many a time, we cannot use the condition of *ceteris paribus*, within science. When everything else is bound to change, these are the only surviving links, which act as a connection between the conditions before and after the change. Plato was clever enough to surmise the fact that symmetry plays an intrinsic role in study of nature. This symmetry has been studied widely in the east and west, be it in the form of sacrificial altars constructed using such symmetrical shapes, or be it the flower of life which was supposed to be the sacred seed of life [9].

Equations describing the fundamental forces of nature, adhere to a beautiful symmetry. For instance, to give you a rough idea, if you compare reality to a room whose shape you have to find out, and the room is so big, thereby it is hard to traverse all the room. Suppose given that it has axis symmetry, you just have to traverse one half of it, and thereby you can get to know the whole picture (since you are guaranteed that, the other half looks the same). This method is more popularly called as Divide and Conquer strategy and it has got interesting applications in the field of computer science as well. Before entering into the latter part, at this stage, I refer the reader to the literature mentioned in the references, for further reading regarding symmetry [10, 11]. Conservation law has been studied within philosophy long ago, before physicists formulated it in a mathematical way. For instance, Aristotle's *De Anima* [12] and Metaphysics [13] bear some of the interesting notions regarding soul, Potentiality[d] and Actuality. The idea of *Entelecheia* (as he calls it), has been taken further, by Leibniz [14], to develop the notion of conservation of energy.

3. Being Invariant and Being Conserved

As we have discussed in the introduction, regarding representation of numbers, I need to add that, there is no objective definition of Energy. As people subscribe to various schools of thoughts (like Formalism, Intuitionism, Conventionalism, etc.), for defining numbers, physicists also subscribe to various forms of it, or define it using epistemic modes. The importance of the theorem lies in the very fact, that it allows us to define such energy. Energy is something that is conserved under time translation. So at-least that's a kind objective definition that one can offer in some sense. We will come to it a little later, and now we will look into the terms conservation and invariance. Going back to the introduction, we have to quantify something before us, before we can define it, epistemically. Descartes introduced reference

[d] I wanted to add at this point, that the notion of Potentiality, helped Heisenberg to formulate his potentia interpretation of quantum states, which to some extent solved measurement problem.

frames, or Cartesian coordinate axes, soon after he realised that one cannot define space, without reference. So we measure the perceivable so called primary properties like Height, Thickness, Mass, etc., of the objects using some calibrated scale. In doing so one obtains a numerical quantity. Invariance refers to that numerical quantity being the same in any reference frame. Conservation on the other hand refers to that quantity, in a given reference frame, remains the same throughout any (given) operation. For instance, take the example of height. I may measure my height using a scale which is calibrated to measure height in feet. I will use it to obtain my height as 6 feet. So if you build a coordinate axes in India, and measure my height, it will turn out to be 6 feet. So if you take me to Germany, and invent a new coordinate system my height will still remain the same, even though the underlying Cartesian coordinates may change (it is vital that they should have the same units). Distance is invariant to linear and rotational transformations. So my height is invariant there, when you take me to a new coordinate system. Given that I am in India, and the **process** considered is aging, my height is bound to change. As I grow I become taller, so it is not **conserved**. Repeating, for a quantity to be invariant the number (we assign) will not change in any coordinate system, where as to be conserved, it should remain the same throughout the process. So if something is invariant it means that some property of it is not changing with respect to given transformations. This can be stated as having symmetry entailed from the given invariance.

4. Getting into "Action"

Let us try to understand the Action Principle. Determinism added a feather to the hat of physics. Newton, Laplace and others figured out equations, that can determine everything in cosmos, and the model goes by the name of clockwork universe. Given initial conditions of a system, physicists are interested in figuring out futuristic evolution of it. All that you need to do is write down Newton's laws of motion. There is a reason, behind Newton choosing $F = m.a$ instead of $F = m.v$. For instance assume that $F = mv$, and v being velocity, which is first derivative of distance with respect to time, $F = m.\frac{dx}{dt} \implies F = m.\ \frac{x_{t+h}-x_t}{h}$, therefore $x_{t+h} = \frac{F.h}{m} + x_t$. So if you know h (infinitesimal change), present position x_t of particle at time t, you can predict the position of particle at some time $t+h$ using the above equation. But you can clearly see that, the equation is not time reversible, meaning, if you replace t with $-t$ the equation $F = m.\frac{dx}{dt}$ will change. So it is not possible to predict the past of the system given the present, but you can predict the future with accuracy. If you drop a stone, from a building, you can easily infer the final position of the stone (i.e. will lie on the ground, if there is no obstruction). But seeing a stone on the ground, you cannot infer from which floor it came, since the final position of the stone is same, regardless whether it is dropped from first floor or hundredth floor. So hence you need a term that bears past in it, which second derivative does. Consider the trajectory of a particle, and when $x(t)$ refers to the position of the particle at time t, x_{t+h}, x_{t-h} (where h is very small) refer to it is future and past positions, respectively. Now, if you consider $F = ma = m\frac{dv}{dt} = m\left\{\frac{\left\{\frac{x_{t+h}-x_t}{h} - \frac{x_t-x_{t-h}}{h}\right\}}{h}\right\}$, you have the term x_{t-h} in it, which $F = mv$ lacks. Hence $F = mv$ is not time reversible. In $F = ma = m\frac{d^2x}{dt^2}$, you can replace t with $-t$ and then see that equation remains the same (second order derivative). So not only you can predict the future but you can reverse the time and predict the past as well. Hence all the physical laws take force as $F = m.a$ and moreover it matches with experimental predictions/observations.

Newton's laws help us, in finding out the futuristic evolution of the system, i.e. you start with the present position, you can calculate the futuristic position, after infinitesimal time, but what if we are given initial position x_i and final position x_f of the particle, how can one predict the trajectory taken by the particle? That is done by using **principle of least action**. In Newtonian physics we study two forms of Energy. One being Kinetic energy, and another is Potential energy, former is related to how fast a body moves, and latter, related to work done by forces on a body. So energy arises as a result of body's velocity and work done on it. If we kick a ball, and we know the initial and final positions, we can use the principle of least action to figure out the trajectory, and can say that among all possible paths connecting initial and final positions, it chooses that path, in which the ball's kinetic energy, minus the potential energy, multiplied with time, is minimum. In other words if you consider the difference between kinetic and potential energies, and integrate them with respect to time, you get something called **action**. So the path with least action is chosen. This property of nature adhering to "least.." should not be taken as it is miserliness, but more of nature being efficient, and we do not know why nature behaves so.[e]

Physicists realised this fact long back, that nature adheres to some kind of least action or least time. For instance Fermat, realised that among all possible paths that a light ray can take, it selects the path that takes least time for it to traverse.[f] Descartes (in his Discourse of method), figured out the principle of least time. Later on after Newton published his Principia, the problem of *brachistochrone* (least time) was studied in a concrete sense. The problem of determining the shape of perimeter, that can hold maximum area within it, goes under the name of Dido's problem (more about it here [17]). The way honey bees construct their honey comb in hexagonal shape, shows us their optimised use of space, that is available before them. They somehow know that such kind of tiling, will fill the whole area, without any gaps, and it is easy to construct as well. Laws as we know are discovered, but the underlying phenomenon are an intricate part of nature, woven into the day to day happenings. For instance, Spider in some sense knows Hooke's law, for verily it weaves its web seeing that stress should be proportional to strain. Since then Huygens *et al.* worked on it, and Euler proposed stationary action along a path, and Hamilton later on founded a concrete law, which now plays a crucial role both in Classical and Quantum physics.

Now comes the fundamental question, how come the ball knows the path it has to take such that it is kinetic energy, minus the potential energy will be less? There are infinite number of paths available before it, and it is not that it is going to visit each path, to sit down and calculate the action and then decide which path to take. The answer comes when you embed the Newton's laws into a bigger structure, called Euler–Lagrangian equations. So once you have Euler–Lagrange equations, you can see how Newton's laws of motion follow from that. So one way to see the success of Newton's laws, lies in the fact that, they are embedded in a bigger structure, which if you work out in a reverse way, makes the action stationary, and thereby selects a path, with least action. So now you can see that particle need not traverse all the possible ways available before it. It has that beautiful Euler–Lagrangian equation which one can apply for any instant of time, which tells the particle how to move, maintaining least action. One may also wonder why only Kinetic and potential energies are involved here. Nature always tries to minimise the difference between the both. If you consider (Newton's) Absolute Space time model, where you can work with Galilean transformations,[g] you can see that there is continuous exchange between kinetic and potential energy. Consider a ball thrown towards the sky, at every point, on ascent, kinetic energy is getting converted into potential energy, and at the top most point, when kinetic energy is zero, balls potential energy again gets converted into kinetic energy. Nature adheres to many such minimum and maximum principles, for instance in an isolated system, entropy is maximum, and so you have such extremals with Gibb's free energy and many other things.

The difference between Kinetic and Potential energy is called as Lagrangian. Let us assume a coordinate system, with coordinates as $\mathfrak{q} = (\mathfrak{q}^1, \ldots, \mathfrak{q}^n)$, then Lagrangian is a function $\mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}, t) = \sum_{i=1}^{n} \frac{1}{2} m(\dot{\mathfrak{q}}^i)^2 - \mathcal{U}(\mathfrak{q}^1, \ldots, \mathfrak{q}^n)$ (i.e. Kinetic Energy minus Potential energy, and we follow Newton's Dot notation for representing a derivative, and hence $\dot{\mathfrak{q}}$ is $\frac{d\mathfrak{q}}{dt}$, $\ddot{\mathfrak{q}} = \frac{d^2\mathfrak{q}}{dt^2}$, etc.). Let us assume that Lagrangian is explicitly time independent (though it depends upon time, implicitly, as coordinates $\mathfrak{q}(t)$ are functions of time). Now the action as we discussed is Lagrangian integrated over time. Given initial position $\mathfrak{q}(t_i)$ and final position $\mathfrak{q}(t_f)$ of the particle at time t_i and t_f respectively, Action $\mathcal{A}(\mathfrak{q}) = \int_{t_i}^{t_f} \mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}})\, dt$. Action $\mathcal{A}(q(t))$ is a complex entity, as it is a function of a function (functional). It is a function of $q(t)$, which in turn is a function of t. So for some set of $t's, q(t)$ produces a family of functions, and when you feed that family of functions to action, it spits out a number. Now among all the paths available between the

[e] Einstein once said, "Nature hides her secrets because of her essential loftiness, but not by means of ruse."

[f] Later on Feynmann formulated his path integral to explain this phenomenon more clearly.

[g] Adding to this, in Einsteinian Space time which is relativistic, this principle of least action plays a geometrical role, in figuring out the geodesics, the shortest path between two points on a non-Euclidean plane, and thereby allowed Einstein to interpret gravity geometrically.

points $\mathfrak{q}(t_i)$, $\mathfrak{q}(t_f)$, that path which minimises the action is chosen. So holding the initial and final points, let us perturb the path little bit such that $\mathfrak{q} \to \mathfrak{q} + \delta\mathfrak{q}$, and the factor $\delta\mathfrak{q}$ (infinitesimal change) which we use to perturb the path, should vanish at end points, as we are not going to disturb them. Hence $\delta\mathfrak{q}(t_i) = \delta\mathfrak{q}(t_f) = 0$. Given a function $g(m_1, \ldots, m_n)$, of n variables, you can vary the variables infinitesimally $m_i \to m_i + \delta m_i$ ($\forall i = 1$ to n), and revamping function after that variation, we obtain $g(m_1 + \delta m_1, \ldots, m_n + \delta m_n)$ which can be related to $g(m_1, \ldots, m_n)$ using taylor expansion as $g(m_1 + \delta m_1, \ldots, m_n + \delta m_n) \approx g(m_1, \ldots, m_n) + \sum_{i=1}^{n} \delta m_i \frac{\partial g}{\partial m_i}$. One assumption we made is that higher order terms $O((\delta m_i)^2)$ are negligible. Now considering action $\mathcal{A}(\mathfrak{q}) = \int_{t_i}^{t_f} \mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}})\, dt$, and under the variation $\mathfrak{q} \to \mathfrak{q} + \delta\mathfrak{q}$, we have

$$\mathcal{A}(\mathfrak{q}+\delta\mathfrak{q}) = \int_{t_i}^{t_f} \mathcal{L}(\mathfrak{q}+\delta\mathfrak{q}, \dot{\mathfrak{q}}+\delta\dot{\mathfrak{q}})\, dt$$

$$\mathcal{A}(\mathfrak{q}+\delta\mathfrak{q}) = \int_{t_i}^{t_f} \left\{ \mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}) + \frac{\partial \mathcal{L}}{\partial \mathfrak{q}}\delta\mathfrak{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\delta\dot{\mathfrak{q}} + O((\delta\mathfrak{q})^2) \right\} dt$$

$$\mathcal{A}(\mathfrak{q}+\delta\mathfrak{q}) = \int_{t_i}^{t_f} \mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}})\, dt + \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}}\delta\mathfrak{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\delta\dot{\mathfrak{q}} \right\} dt$$

$$\mathcal{A}(\mathfrak{q}+\delta\mathfrak{q}) = \mathcal{A}(\mathfrak{q}) + \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}}\delta\mathfrak{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\delta\dot{\mathfrak{q}} \right\} dt$$

$$\mathcal{A}(\mathfrak{q}+\delta\mathfrak{q}) - \mathcal{A}(\mathfrak{q}) = \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}}\delta\mathfrak{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\delta\dot{\mathfrak{q}} \right\} dt.$$

From integration by parts, if we have two functions $f(x), g(x)$, and $F(x) = f(x)g(x)$, thereby $\frac{dF}{dx} = f\dot{g} + g\dot{f}$, and so $\int_{x_1}^{x_2} \frac{dF}{dx}\, dx = \int_{x_1}^{x_2} f\dot{g}\, dx + \int_{x_1}^{x_2} \dot{f}g\, dx$, where $\int_{x_1}^{x_2} \frac{dF}{dx}\, dx = F(x)|_{x_1}^{x_2}$. So by that, you can write $\int_{t_i}^{t_f} \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}} \frac{d}{dt}(\delta\mathfrak{q})\, dt = -\int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\right)\delta\mathfrak{q}\, dt + \left[\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\delta\mathfrak{q}\right]_{t_i}^{t_f}$, and the last term vanishes since change at end points, is zero.

$$\mathcal{A}(\mathfrak{q}+\delta\mathfrak{q}) - \mathcal{A}(\mathfrak{q}) = \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}}\delta\mathfrak{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\frac{d}{dt}(\delta\mathfrak{q}) \right\} dt$$

$$\delta\mathcal{A}(\mathfrak{q}) = \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}}\delta\mathfrak{q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\right)\delta\mathfrak{q} \right\} dt$$

$$\delta\mathcal{A}(\mathfrak{q}) = \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\right) \right\} \delta\mathfrak{q}\, dt$$

$$\frac{\delta\mathcal{A}(\mathfrak{q})}{\delta\mathfrak{q}} = \int_{t_i}^{t_f} \left\{ \frac{\partial \mathcal{L}}{\partial \mathfrak{q}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\right) \right\} dt.$$

Action is extremised if, $\frac{\delta\mathcal{A}(\mathfrak{q})}{\delta\mathfrak{q}} \to 0$ so which implies that integrand on RHS should be zero. Which is nothing but saying $\frac{\partial \mathcal{L}}{\partial \mathfrak{q}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}\right)$. So that equation we obtained is called Euler–Lagrangian Equation (E.L.E), which encapsulates many things related to motion. For instance you can derive Newton's laws of motion, taking Lagrangian for one dimensional motion, as $\mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}) = \frac{1}{2}m\dot{\mathfrak{q}}^2 - \mathcal{U}(\mathfrak{q})$, and you feed it into E.L.E, you will get $m\ddot{\mathfrak{q}} = -\frac{d\ \mathcal{U}(\mathfrak{q})}{d\mathfrak{q}}$ which is Newton's law ($\ddot{\mathfrak{q}}$ being the acceleration). You can also see that the term $\frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}}$ gives out $m\dot{\mathfrak{q}}$ which is momentum. So generally, $p_i = \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}^i}$ is called the conjugate momentum.

5. Conservation Laws

So as we have discussed, invariance is an important property. For instance, you go to a shop and buy a costly watch. You want it to remain invariant (in context of working flawlessly, as it used to work inside the shop) for a long time. But since such invariance is not guaranteed, companies promise that goods will be invariant, and will not be spoiled, for a particular time period (so called guaranty and warranty period). But you cannot expect universe to offer you a set of laws with a warranty period. That would be as if Higgs field is giving you a warranty that you will have same amount of mass, till 3 years, and after that your mass is bound to increase and decrease.[h] So Noether's wonderful insight was to link up symmetries with conserved quantities. We will come to the profound implications of that theorem.

Recall those school days. You have done an experiment in the beginning of the year, and recorded the results of it. You have got to repeat the same experiment in the term exams, and you will obtain the same results, if you repeat the experiment in an exact way, under the same conditions. We can now say that your experiment is invariant over time, if you do it today, or some other day, it will give you same results. So something should be conserved. Though if you assume that Lagrangian is time independent, as we have seen it has implicit time dependence through coordinate variables which depend upon time. Consider $\frac{d\mathcal{L}(\mathfrak{q}^i, \dot{\mathfrak{q}}^i)}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial \mathfrak{q}^i}\dot{\mathfrak{q}}^i + \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}^i}\ddot{\mathfrak{q}}^i$, and we know that $p_i = \frac{\partial \mathcal{L}}{\partial \dot{\mathfrak{q}}^i}$, we can write $\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial \mathfrak{q}^i}$ (using Euler–Lagrangian Equations). Now, we can write $\frac{d\mathcal{L}}{dt} = \sum_i \dot{p}_i\dot{\mathfrak{q}}^i + p_i\ddot{\mathfrak{q}}^i$. By chain rule you can write, RHS as $\frac{d}{dt}\left(\sum_i p_i\dot{\mathfrak{q}}^i\right)$ and now you can see

[h]May be a good news for those who are obese, but not for all.

that though Lagrangian may not be conserved, but this quantity, $\frac{d}{dt}\left(\sum_i p_i \dot{q}^i\right) - \frac{d\mathcal{L}}{dt} = 0$ which is $\frac{d}{dt}\left\{\left(\sum_i p_i \dot{q}^i\right) - \mathcal{L}\right\}$ is conserved. So this very term $\left(\sum_i p_i \dot{q}^i\right) - \mathcal{L}$ has a special name, and it is called the Hamiltonian, or the total energy of a system. So if Lagrangian is time invariant, then Total energy, or Hamiltonian is conserved (more about Hamiltonian refer to Appendix), and this is so called **Law of Conservation of Energy**.

Now imagine you are playing pocket billiards in a big room sans windows, and you have no clue of what is happening outside the room. Assume that I start pulling the carpet on which you are standing slowly, and since you are so engrossed in playing, you do not notice that you are moving. After sometime, you are shifted to a new position within the same room. Apropos, you will still continue to play in the same way. You will get the same results, and you are spatially invariant, since you do not even notice any change. So in the same way, I start rotating the whole room, and still you continue to play, in the same way. After sometime you are at an angle of say 35° with respect to the horizontal axis. It will not make any change, since I am moving the whole setup, and you are rotationally invariant. But what if, I suddenly press a button, which pulls the room down. Then you will fall down, and upon hitting the ground you give an angry *what-happened* look, scratching your head. So any such sharp changes in velocity ruin invariance. Your velocity, changed, and all the billiard balls got disturbed. Now you have to arrange them again using a rack. So you are invariant of spatial and angular transformations, and there will be two associated quantities corresponding to them.

Let us assume that $q^i = (q^1, q^2)$ (you can consider i running from 1 to any N, and work out in a similar way, but I am considering two dimensions) denote your position in the room, and I displaced the whole setup, in the direction of $k^i = (k^1, k^2)$ through a distance of b. So now the new coordinates will be $q^i = q^i + b\ k^i$. Since we assumed that Lagrangian is invariant of such transformations, we can say that $\left.\frac{d\mathcal{L}}{db}\right|_{b=0} = 0$, and it follows that $\frac{dq^i}{db} = k^i$, and since we shift your position so slowly, without changing velocity, we can say that $\frac{d\dot{q}^i}{db} = 0$. Now considering that $0 = \frac{d\mathcal{L}}{db} = \sum_{i=1}^{2} \frac{\partial \mathcal{L}}{\partial q^i}\frac{dq^i}{db} + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\frac{d\dot{q}^i}{db}$, and since the second term vanishes, and plugging in k^1 and k^2, $0 = \frac{\partial \mathcal{L}}{\partial q^1}k^1 + \frac{\partial \mathcal{L}}{\partial q^2}k^2$ and by Euler–Lagrangian equation we can write for all i, $\frac{\partial \mathcal{L}}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)$ and so we can write $0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^1}\right)k^1 + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^2}\right)k^2$ and *en passant* k^1, k^2 choosen arbitrarily, to satisfy that equation, both $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^1}\right)$, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^2}\right)$ must be zero. So we can see that $p_1 = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^1}\right)$ and $p_2 = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^2}\right)$ are components of momentum corresponding to q^1, q^2 respectively, and $\frac{dp_1}{dt} = 0, \frac{dp_2}{dt} = 0$, which means that momentum remains unchanged (conserved) over time. The vector $P = (p^1, p^2)$ can be visualised as the total linear momentum. This is **Law of Conservation of Linear Momentum**.

Now let us consider the case of rotations. To rotate coordinates we take the help of rotation matrix R, and if we work in two dimensions (q^1, q^2), the rotation operation is matrix multiplication which is

$$\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$$

such that $q^1 \to q^1\cos\theta - q^2\sin\theta$, $q^2 \to q^1\sin\theta + q^2\cos\theta$, for very small ϵ, if we take $\theta = \epsilon$ we can write $\cos\epsilon \approx 1$ and $\sin\epsilon \approx \epsilon$ when $\epsilon \to 0$ (using small angle approximations). Now substituting that, by rotation coordinates transform as $q^1 \to q^1 - \epsilon q^2, q^2 = q^2 + \epsilon q^1$, thereby $\delta q^1 = -\epsilon q^2, \delta q^2 = \epsilon q^1$. Now for a change, let us ask this question, given a function $\mathcal{L}(q^i, \dot{q}^i)$ what is $\delta\mathcal{L}$, in other words what is the variation in $\mathcal{L}$, when we vary its parameters a little. $\delta\mathcal{L} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial q^i}\delta q^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\delta \dot{q}^i\right]$, using Euler–Lagrangian Equations, and conjugate momentum p_i, we can write it as $\left[\sum_i \dot{p}_i \delta q^i + p_i \delta \dot{q}^i\right]$ which is nothing but, $\frac{d}{dt}\sum_i \left[p_i \delta q^i\right]$. So now under the variation δ if $\delta\mathcal{L} = 0$ then $\frac{d}{dt}\sum_i \left[p_i \delta q^i\right] = 0$ (which is called Noether's Charge). Now substituting the obtained values for δq^i (which are $\delta q^1 = -\epsilon q^2, \delta q^2 = \epsilon q^1$), we obtain $\frac{\delta q^1}{d\epsilon} = -q^2, \frac{\delta q^2}{d\epsilon} = q^1$). Since $\delta\mathcal{L} = \frac{d}{dt}\sum_i \left[p_i \delta q^i\right] = 0$, we can consider the parameter as ϵ and since we assume that Lagrangian is invariant of ϵ, $\frac{d\mathcal{L}}{d\epsilon} = \sum_i \left[p_i \frac{dq^i}{d\epsilon}\right] = 0$. So if Lagrangian is invariant over that rotation ϵ then it means the conserved quantity is $\sum_i \left[p_i \frac{dq^i}{d\epsilon}\right]$, and since we are working with two dimensions here, it is $p_1 \frac{dq^1}{d\epsilon} + p_2 \frac{dq^2}{d\epsilon}$, and substituting the values we have we can write the quantity that is conserved is $p_2 q^1 - p_1 q^2$. The term is called Angular momentum L about the third axis, other than q^1 and q^2. This pretty much summarises **Conservation Law for Angular Momentum**.

6. Mathematical Codification of Noether's Theorem, for Aficionados

By a configuration space, I mean a space whose points correspond to all possible configurations (Degrees of Freedom) a system (say a rigid body or a group of particles) can occupy. One would naturally jettison, the idea of using Euclidean space, and go for a manifold, to account complicated configurations, since a latter has got non trivial topology, which former lacks. Additionally a manifold resembles Euclidean space locally. A curve in such configuration space represents the futuristic evolution of the system. Globally, a Manifold need not be flat, and thereby it is hard to generalise the notion of vector *en scène*. So one would naturally construct a tangent space at every point on that manifold, which bears all possible vectors pointing tangentially with respect to that point. Now on that tangent space, one can introduce the notions of addition, subtraction, transformation of vectors, etc., and all possible velocities that the system can take, are represented as vectors, which are housed in the tangent space, corresponding to its configuration space.

In a given configuration space, say $\mathfrak{M}$ which is a manifold, the points are labelled by $\mathfrak{q}^i$ (generalised coordinates $i = 1, 2, \ldots, N$ and N being Degrees of freedom) which represent the possible positions a system can occupy. Velocity $\dot{\mathfrak{q}}^i$ (dot represents the first time derivative) of $\mathfrak{q}^i$ (function of t), is a tangent vector lying in the tangent space $\mathcal{T}_{\mathfrak{q}^i}\mathfrak{M}$ corresponding to $\mathfrak{M}$. Given a set of coordinates $\mathfrak{q}^i$ tangent space $\mathcal{T}_{\mathfrak{q}^i}\mathfrak{M}$ is spanned by the basis of tangent vectors at zero, $\left\{\frac{\partial}{\partial \mathfrak{q}^i}\right\}, \forall i$. Subsequently, any (velocity) tangent vector on tangent space, $\dot{\mathfrak{q}}$ with values $\dot{\mathfrak{q}}^i$ can be expressed as a linear combination of basis tangent vectors as, $\dot{\mathfrak{q}} = \sum_{\forall i} \dot{\mathfrak{q}}^i \frac{\partial}{\partial \mathfrak{q}^i}$. At this point one would prefer bundling up all such tangent spaces, to obtain tangent bundle $\mathcal{T}\mathfrak{M}$, by taking a disjoint union on tangent spaces, i.e.

$$\mathcal{T}\mathfrak{M} = \bigcup_{\mathfrak{q}^i \in \mathfrak{M}} \mathcal{T}_{\mathfrak{q}^i}\mathfrak{M} = \bigcup_{\mathfrak{q}^i \in \mathfrak{M}} \left\{ \left(\mathfrak{q}^i, \dot{\mathfrak{q}}^i\right) \middle| \dot{\mathfrak{q}}^i \in \mathcal{T}_{\mathfrak{q}^i}\mathfrak{M} \right\}.$$

The Fiber Coordinates $\left(\mathfrak{q}^i, \dot{\mathfrak{q}}^i\right)$ of the tangent bundle $\mathcal{T}\mathfrak{M}$ adhere to a natural projection $\pi : \mathcal{T}\mathfrak{M} \mapsto \mathfrak{M}$, such that $\pi\left(\mathfrak{q}^i, \dot{\mathfrak{q}}^i\right) = \mathfrak{q}^i$, which thereby maps each tangent space $\mathcal{T}_{\mathfrak{q}^i}\mathfrak{M}$ onto a point $\mathfrak{q}^i$ on the manifold $\mathfrak{M}$. It is not far to see that tangent bundle is, but a kind of vector bundle. By a **path** or **trajectory** I mean a function $\mathfrak{q} : [t_a, t_b] \mapsto \mathfrak{M}$, such that at every instant of time t_m, $\mathfrak{q}(t_m) = \mathfrak{q}^k$, spits out the position that a system takes in configuration space, corresponding to that time. Given two fixed end points $\mathfrak{q}^a$ and $\mathfrak{q}^b$ in a configuration space, there are many paths connecting them. Trajectory space $\xi_{\mathfrak{M}}(\mathfrak{q}^a, \mathfrak{q}^b)$, is a set that captures all possible paths, between those end points, i.e. $\xi_{\mathfrak{M}}(\mathfrak{q}^a, \mathfrak{q}^b) = \left\{\mathfrak{q} : [t_a, t_b] \mapsto \mathfrak{M} \,\middle|\, \mathfrak{q}(t_a) = \mathfrak{q}^a, \mathfrak{q}(t_b) = \mathfrak{q}^b\right\}$. To add, $\xi_{\mathfrak{M}} = \left\{\xi_{\mathfrak{M}}(\mathfrak{q}^a, \mathfrak{q}^b)\right\} \forall\ \mathfrak{q}^a, \mathfrak{q}^b \in \mathfrak{M}$ is an infinite dimensional manifold. Now, we can define Lagrangian to be a function $\mathcal{L} : \mathcal{T}\mathfrak{M} \mapsto \mathbb{R}$.[i] On the local coordinates of the Tangent Bundle, $\mathcal{L}(\mathfrak{q}^1, \ldots, \mathfrak{q}^N, \dot{\mathfrak{q}}^1, \ldots, \dot{\mathfrak{q}}^N) = \frac{1}{2} m g_{ij} \dot{\mathfrak{q}}^i \dot{\mathfrak{q}}^j - \mathcal{U}(\mathfrak{q}^1, \ldots, \mathfrak{q}^N)$, which is nothing but the difference between kinetic and potential energies of the system, and following Einstein's summation convention g_{ij} is a metric tensor, and kinetic energy, is summed up over all coordinates. Given initial and final positions $\mathfrak{q}^a, \mathfrak{q}^b \in \mathfrak{M}$, one can define an **action** as $\mathcal{A} : \xi_{\mathfrak{M}}(\mathfrak{q}^a, \mathfrak{q}^b) \mapsto \mathbb{R}$, $\mathcal{A}(\mathfrak{q}) = \int_{t_a}^{t_b} \mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}, t) dt$, where ξ is defined as above. Amongst all paths in ξ nature chooses that path for which $\mathcal{A}(q)$ is minimum.

Given a trajectory space, $\xi_{\mathfrak{M}}(\mathfrak{q}^a, \mathfrak{q}^b)$, and a path $\mathfrak{q}$ in it, with end points, $\mathfrak{q}^a, \mathfrak{q}^b$ one can consider (one parameter) smooth map $\Psi : \xi_{\mathfrak{M}} \times \mathbb{R} \mapsto \xi_{\mathfrak{M}}$, with the properties, $\Psi_0(\mathfrak{q}) = \mathfrak{q}$ is an identity map, $\Psi_s(\mathfrak{q}) = \mathfrak{q}_s$ for some $s \in \mathbb{R}$, and $\mathfrak{q}_s$ is a new path in the space $\xi_{\mathfrak{M}}$ with end points $\Psi(\mathfrak{q}^a), \Psi(\mathfrak{q}^b)$. More importantly, Ψ satisfies $\Psi_p \circ \Psi_s(\mathfrak{q}) = \Psi_{s+p}(\mathfrak{q}) = \mathfrak{q}_{s+p}$. Suppose given that path $\mathfrak{q}$ extremises $\mathcal{A}(\mathfrak{q})$, we cannot assure that $\mathfrak{q}_s$ (which is $\Psi_s(\mathfrak{q})$) necessarily makes the action stationary, unless the Lagrangian is invariant, i.e. $\mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}) = \mathcal{L}(\Psi_s(\mathfrak{q}), \Psi_s(\dot{\mathfrak{q}}))$, where $\Psi_s(\dot{\mathfrak{q}}) = \dfrac{d}{dt}\Psi_s(\mathfrak{q})$. So now starting with $\mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}) = \mathcal{L}(\Psi_s(\mathfrak{q}), \Psi_s(\dot{\mathfrak{q}})) \Rightarrow \dfrac{d}{ds}\mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}}) = \dfrac{d}{ds}\mathcal{L}(\Psi_s(\mathfrak{q}), \Psi_s(\dot{\mathfrak{q}}))$. Since *LHS* do not depend upon s, using chain rule

$$0 = \frac{d}{ds}\mathcal{L}(\Psi_s(\mathfrak{q}), \Psi_s(\dot{\mathfrak{q}}))$$

$$0 = \frac{d\mathcal{L}}{d\Psi_s(\mathfrak{q})} \frac{d\Psi_s(\mathfrak{q})}{ds} + \frac{d\mathcal{L}}{d\Psi_s(\dot{\mathfrak{q}})} \frac{d}{ds}\left(\frac{d}{dt}(\Psi_s(\mathfrak{q}))\right)$$

$$\left(\because \Psi_s(\dot{\mathfrak{q}}) = \frac{d}{dt}\Psi_s(\mathfrak{q})\right)$$

By using Euler Lagrangian Equation

$$\frac{d\mathcal{L}}{d\Psi_s(\mathfrak{q})} = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\Psi_s(\dot{\mathfrak{q}})}\right)$$

[i] If Lagrangian has explicit time dependence, then $\mathcal{L} : \mathcal{T}\mathfrak{M} \times \mathbb{R} \mapsto \mathbb{R}$.

we can rewrite the above equation as

$$0 = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\Psi_s(\dot{\mathfrak{q}})}\right)\frac{d\Psi_s(\mathfrak{q})}{ds} + \frac{d\mathcal{L}}{d\Psi_s(\dot{\mathfrak{q}})}\frac{d}{dt}\left(\frac{d\Psi_s(\mathfrak{q})}{ds}\right)$$

$$0 = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\Psi_s(\dot{\mathfrak{q}})}\frac{d\Psi_s(\mathfrak{q})}{ds}\right)\text{[Chain rule]}$$

You can see that term in the brackets, which does not change with time, and it is called the constant of motion. You can go further and evaluate it at $s = 0$, and since it is identity element $\Psi_s(\mathfrak{q})|_{s=0} = \mathfrak{q}$, and so the constant of the motion is $\frac{d\mathcal{L}}{d\left(\frac{d\mathfrak{q}}{dt}\right)}\frac{d\Psi_s(\mathfrak{q})}{ds}\Big|_{s=0}$, which is $\frac{d\mathcal{L}}{d\dot{\mathfrak{q}}}\frac{d\Psi_s(\mathfrak{q})}{ds}\Big|_{s=0}$. Noether's theorem, can be stated as, "*Given a set of transformations $\{\Psi_s\}$ under which Lagrangian is invariant, then such an invariance entails a conserved quantity $\frac{d\mathcal{L}}{d\dot{\mathfrak{q}}}\frac{d\Psi_s(\mathfrak{q})}{ds}\Big|_{s=0}$, which is constant of motion, which remains unchanged*. Now, at this point, for the sake of brevity, consider Lagrangian of a free particle as $\mathcal{L} = \frac{1}{2}m\dot{\mathfrak{q}}^2$, if you are *au fait* with Vector calculus and Lie groups, it is not far to see that for the transformations $\Psi_s(\mathfrak{q}(t)) = \mathfrak{q}(s+t)$, $\Psi_s(\mathfrak{q}(t)) = \mathfrak{q}(t)+sk$, $\Psi_s(\mathfrak{q}(t)) = e^{s\mathbb{M}}\mathfrak{q}(t)$ the conserved quantities are $\sum_{i=1}^{N} p^i\dot{\mathfrak{q}}^i - L$ (Hamiltonian), $\sum_{i=1}^{N} m\dot{\mathfrak{q}}^i.k$ (Linear Momentum along vector k) and $\sum_{i=1}^{N} m\dot{\mathfrak{q}}^i(\mathbb{M}.\mathfrak{q})^i$ (Total Angular Momentum) respectively, where $\mathfrak{q}(t)$ refer to generalised coordinates as a function of time, k is an element of vector space, and $\mathbb{M}$ (anti-symmetric matrix) belongs to Lie Algebra $\mathfrak{so}(N)$ of the corresponding Lie group $SO(N)$ ($e^{s\mathbb{M}} \in SO(N)$ is a rotation matrix). This pretty much summarises the mathematical formulation of the theorem.

7. Conclusion

If Lagrangian is invariant of certain transformations, it has a symmetry, and thereby there is a conserved quantity anent to such symmetry. Apart from establishing one to one correspondence between the conserved quantities, and symmetries underlying the Lagrangian, Noether's theorem has far reaching implications. One can see that it actually connects the epistemic and ontic states in some sense. Einstein's conception of space and time being relative, reflects the fact that they have a subjective basis, and one can obtain the same experimental results, by synchronising their clocks or scaling their coordinates accordingly between reference frames. On the other hand quiddities.[j] like Energy, momentum, etc., have ontological basis, but can be subjectively perceived as well. If a theory is invariant of subjectively perceived (epistemic data, such as distance and time) constructs, it has a corresponding conserved entity (ontological, like energy and momentum), verily captures the essence of the Noether's theorem from a philosophical perspective, apart from it is formal definition. I conclude this article, with this verse of *Bhartṛhari*.[k] who in his *Nitishatakam*, referring the eternal victory attained by masters, says,"Nāsti yeṣāṁ yaṣāḥ kāye jarāmaraṇajaṁ bhayaṁ", "Whose body of fame, has no fear of age or death". Noether, as mentioned above, continues to live forever, through her works.

8. Appendix

This energy has a special name within physics. It is called Hamiltonian of a system, which is the total energy of the system, denoted by $\mathcal{H} = K.E + P.E$, you may then wonder what's the use of Hamiltonian when we have Lagrangian with us? Hamiltonian is very important in quantum mechanics as it has got a symplectic structure. Analogous to the Euler's–Lagrangian equations, we have two intertwined equations namely

$$\dot{\mathfrak{q}}^i = \frac{\partial\mathcal{H}}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial\mathfrak{q}^i}.$$

$\mathfrak{q}^i$'s and p_i's are generalised coordinates for more than one dimension, and $\dot{\mathfrak{q}}$ and $\dot{p}$ being first order derivatives of position and momentum with respect to time. These two equations are called as Hamiltonian equations. So using both of them, you can predict the evolution of the system, in phase space. More importantly, you can model the motion in phase space, as a fluid, using a vector field. Hamiltonian is a function of position $\mathfrak{q}$ and momentum p, denoted by $\mathcal{H}(\mathfrak{q}, p)$, where as Lagrangian is a function of position q and velocity $\dot{\mathfrak{q}}$, denoted by $\mathcal{L}(\mathfrak{q}, \dot{\mathfrak{q}})$. Hamiltonian is a constant of motion, it wont change over time (it is conserved), where as Lagrangian itself is not conserved, but it has conserved quantities corresponding to each

[j]Derived from Latin, *Quidditas*, which means the essence of object/their universal qualities (whatness).

[k]A Hindu Philosopher, Poet and a grammarian, who lived around 5CE, is well known for his magnificent works in Sanskrit. Nitishataka, is a set of 100 poems written on human behaviour in a civilised society, on polity prudence and wisdom.

invariant it has, which follow from Noether's theorem. You can construct action principle from Hamiltonian as well, and I just leave it here by saying that, you can convert Lagrangian to Hamiltonian using Legendre transforms.

Acknowledgements

Author gratefully acknowledges the help of Professor Sujatha Ramdorai, in reading through the penultimate draft of the paper, and offering valuable comments. Author thanks Mr Abhiram M K for commenting on the paper at various stages and Mr M G Subramanian for his feedback. Author thanks Mrs Avnita Bir and others at R N Podar School, for being kind towards him. Author finally thanks Professor Sridharan, for inspiring him through his writings and conversations. The full responsibility for the paper, naturally rests solely with the author.

Dedicated to that self luminous omnipresent radiance, which directs my understanding, and ensures my well being!

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