The Howe Duality Conjecture

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1. Introduction

This article is concerned with a conjecture in representation theory which was formulated by Roger E Howe [8] of Yale University in the mid-1970s. Let us briefly recall the basic objects of representation theory.

1.1. *Representation theory*

Representation theory was created at the end of the 19th century through the work of F G Frobenius (1849–1917) and I Schur (1875–1941). Recall that groups often occur in nature not as an abstract entity but as the set of symmetries of some natural object X, i.e. $G \cong \mathrm{Aut}(X)$. From this point of view, a representation ρ of a group G on a vector space V (over the field $\mathbb{C}$ of complex numbers, say) is an action of G on V as linear transformations, i.e. a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V).$$

Thus representation theory involves an interplay between group theory and linear algebra. Any representation ρ as above is irreducible if the only G-invariant subspaces are the zero subspace or the whole space V. When the representation theory of G is nice (for example when one considers finite groups G and finite-dimensional representations over $\mathbb{C}$), any representation ρ can be decomposed into the direct sum of irreducible subrepresentations. Hence the basic objects are the irreducible representations, and a basic question is the classification of the set $\mathrm{Irr}\,(G)$ of the isomorphism classes of irreducible representations of G.

1.2. *The problem*

In representation theory, one often encounters the following situation: a pair of groups G and H act naturally (as linear transformations) on some vector space V and the actions of G and H commute with each other. In other words, one has a representation of the direct product group $G \times H$ on V. In this case, one may try to decompose V into its irreducible components as a representation of $G \times H$. One may write this decomposition as follows:

$$V = \bigoplus_{\pi \in \mathrm{Irr}\, G} \pi \otimes \Theta(\pi)$$

where the sum runs over irreducible representations π of G and $\Theta(\pi)$ is the multiplicity space, which inherits an action of H. This is analogous to the simultaneous diagonalisation of a commuting family of diagonalisable matrices. Thus $\Theta(\pi)$ is a (possibly zero and possibly reducible) representation of H. One can then raise the following questions:

(a) Is it the case that $\Theta(\pi)$ is irreducible or zero? If so, Θ defines a map

$$\Theta : \mathrm{Irr}\,(G) \longrightarrow \mathrm{Irr}\,(H) \cup \{0\}.$$

(b) Can one determine precisely when $\Theta(\pi)$ is nonzero?

(c) Can one describe $\Theta(\pi)$ (if it is nonzero) in terms of π (somehow)?

Question (c) above would make sense if one knows some sort of classification of $\mathrm{Irr}\,(G)$ and $\mathrm{Irr}\,(H)$: one can then formulate an answer in terms of this classification. When such a classification is not known, one can view the map Θ as giving a construction of (some) irreducible representations of H starting from those of G.

2. Some Examples

Let me give some examples of the above scenario:

2.1. *Schur–Weyl duality*

Let V be a complex vector space and N a positive integer. Then one has a natural representation of $S_N \times \mathrm{GL}(V)$ on $V^{\otimes N}$. The decomposition of this into its irreducible components is the theory of Schur–Weyl duality, which relates the representation theory of the symmetric groups S_N with those of general linear groups $\mathrm{GL}(V)$. This theory was developed in the 1939 classic book [15] of H Weyl (1885–1955) and continues to be an active area of research today.

2.2. *Deligne–Lusztig theory*

Let G be a finite group of Lie type over a finite field $\mathbb{F}_q$ with q elements and let $T \subset G$ be a maximal torus. P Deligne and G Lusztig constructed in [1] a variety X over $\mathbb{F}_q$ on which $T(\mathbb{F}_q) \times G(\mathbb{F}_q)$ acts naturally; X is the so-called Deligne–Lusztig variety. Considering the induced action on (l-adic) cohomology, one has a natural representation of $T(\mathbb{F}_q) \times G(\mathbb{F}_q)$ on $H^*(X(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_l)$. The decomposition of this gives a construction of elements of $\mathrm{Irr}\,(G(\mathbb{F}_q))$ in terms of $\mathrm{Irr}\,(T(\mathbb{F}_q))$. This construction is a basic step for the classification of $\mathrm{Irr}\,(G(\mathbb{F}_q))$ due to Lusztig and others.

2.3. *Local Langlands correspondence*

Let $\mathbb{Q}_p$ be the field of p-adic numbers and let $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be its absolute Galois group. It contains a dense subgroup $W_{\mathbb{Q}_p}$ called the Weil group. It turns out that there is a natural action of $\mathrm{GL}_n(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ on the cohomology $H^*(\mathfrak{X})$ of a rigid analytic variety $\mathfrak{X}$ (the so-called Lubin–Tate space). In decomposing this, one gets a bijection

$$\mathrm{Irr}_{sc}(\mathrm{GL}_n(\mathbb{Q}_p)) \longleftrightarrow \mathrm{Irr}_n(W_{\mathbb{Q}_p})$$

in the style of (a) above. Here, the subscript sc on the left refers to the subset of supercuspidal representations, whereas the n on the right refers to n-dimensional representations of $W_{\mathbb{Q}_p}$. This map is the key ingredient in the proof of the local Langlands conjecture for GL_n by M Harris and R Taylor [6] (with subsequent simpler proofs by G Henniart [7] and P Scholze [11]). Thus, the local Langlands conjecture is nothing but a classification of $\mathrm{Irr}\,(\mathrm{GL}_n(\mathbb{Q}_p))$ in terms of the representation theory of $W_{\mathbb{Q}_p}$.

3. Theta Corrrespondence

In this article, we want to describe another instance of the above scenario: the theory of theta correspondence. As the name suggests, this area has its origins in the theory of theta functions which were discovered by C G J Jacobi (1804–1851) in the first half of the 19th century.

3.1. *Weil representations*

In an influential paper [14] in 1964, A Weil (1906–1998) interpreted theta functions in the framework of representation theory. More precisely, let us work over a characteristic 0 local field (i.e. $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$) henceforth. Then Weil interpreted (the local analog of) theta functions as vectors in an infinite-dimensional representation Ω of a so-called metaplectic group $\mathrm{Mp}(W)$, which is a double cover of a symplectic group $\mathrm{Sp}(W)$ associated to a symplectic vector space W over F. The representation Ω is typically called the Weil representation or the oscillator representation. The terminology "oscillator" refers to the fact that the Weil representation occurs naturally in the quantum mechanical description of the simple harmonic oscillator.

3.2. *Dual pairs*

In the mid-1970s, Howe [8] introduced the notion of dual pairs in $\mathrm{Mp}(W)$: these are subgroups of $\mathrm{Mp}(W)$ of the form $G \times H$ where G and H are mutual centralisers of each other. He gave a classification and construction of all such possible dual pairs. They basically take the following form:

(i) if U is a quadratic space with corresponding orthogonal group $\mathrm{O}(U)$ and V a symplectic space with corresponding metaplectic group $\mathrm{Mp}(V)$, then $W = U \otimes V$ is naturally a symplectic space, and $\mathrm{O}(U) \times \mathrm{Mp}(V)$ is a dual pair in $\mathrm{Mp}(W) = \mathrm{Mp}(U \otimes V)$.

(ii) $\mathrm{U}(V) \times \mathrm{U}(V')$, where V and V' are Hermitian and skew-Hermitian spaces respectively for a quadratic extension E/F.

(iii) $\mathrm{GL}(U) \times \mathrm{GL}(V)$, where U and V are vector spaces over F.

The dual pairs in (i) and (ii) are called Type I dual pairs, while those in (iii) are called Type II. It is particularly easy to describe the Weil representation Ω for Type II dual pairs. The group $\mathrm{GL}(U) \times \mathrm{GL}(V)$ acts naturally on $U \otimes V$ and hence on the space $\mathcal{S}(U \otimes V)$ of Schwarz functions: this is the Weil representation Ω.

3.3. *Theta correspondence*

Howe then considered the restriction of the Weil representation Ω of $\mathrm{Mp}(W)$ to $G \times H$. Writing

$$\Omega|_{G\times H} = \bigoplus_{\pi \in \mathrm{Irr}\,(G)} \pi \otimes \Theta(\pi),$$

we see that each $\Theta(\pi)$ is a representation of H. It was shown by Kudla [9] that $\Theta(\pi)$ is a finite length representation and so one may consider its maximal semisimple quotient (the cosocle) $\theta(\pi)$. We can now formulate:

Howe Duality Conjecture: For any $\pi \in \mathrm{Irr}\,(G)$, $\theta(\pi)$ is either zero or irreducible.

This somewhat innocuous-looking statement is quite fundamental, as it predicts a close relation between the representation theory of G and H. The map

$$\theta : \mathrm{Irr}\,(G) \longrightarrow \mathrm{Irr}\,(H) \cup \{0\}$$

is called the local theta correspondence.

We recall some progress towards this conjecture since it was first conceived:

- In [8], Howe proved the conjecture over $\mathbb{R}$ and $\mathbb{C}$, as well as special cases over p-adic fields.
- Waldspurger [13] showed in 1990 that the conjecture holds for all p-adic fields when $p \neq 2$.
- Kudla [9] showed in 1986 that the conjecture holds for supercuspidal representations over any p-adic field.
- In 2008, Minguez [10] proved the Howe duality conjecture for Type II dual pairs.

Thus, one almost knows the truth of the conjecture, but the missing case $p = 2$ is a nuisance in applications. Moreover, the existing proof of Waldspurger [13] is very complicated and long. Many experts believe, however, that this basic conjecture should have a simple proof. In June 2014, W T Gan and S Takeda [5] succeeded in giving a short and simple proof (10 pages) of the Howe duality conjecture for all p-adic fields, inspired by Minguez's proof [10] in the Type II case.

3.4. *Other questions*

The Howe duality conjecture deals with Question (a) formulated at the beginning of this article. Let me close with some recent sample results concerning Questions (b) and (c), especially by my colleagues in National University of Singapore:

- In a recent paper [12] by B Y Sun and C B Zhu, a conjecture of Kudla and Rallis (the so-called conservation relation conjecture) concerning Question (b) was resolved.
- In joint work [2, 3] of A Ichino and W T Gan, Question (c) was fully addressed for dual pairs of almost equal size, in terms of the local Langlands correspondence.
- In a recent work, H Y Loke and J J Ma resolved Question (c) when π is an epipelagic supercuspidal representation.

The local theta correspondence has many interesting applications. As an example, it is the main tool used in the paper [4] to establish the local Langlands conjecture for GSp(4) and in [3] to establish the remaining cases of the so-called local Gross–Prasad conjecture.

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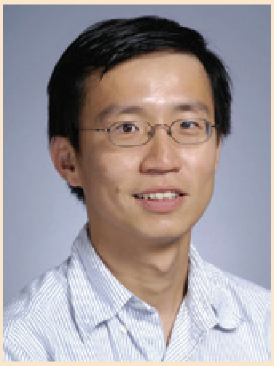

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