Classical Reciprocity Laws

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Abstract. Using the quadratic reciprocity law as the motivating example, we convey an understanding of classical reciprocity laws.

Ever since Gauß published his *Disquisitiones* in 1801, reciprocity laws have been one of the main preoccupations of arithmeticians. My purpose here is not to go into the history of these laws but to convey our present understanding. The adjective *classical* refers to the fact that we assume the relevant roots of unity are present in the number field under discussion.

Let us begin with the quadratic reciprocity law. For every prime number p, we have the finite field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Its multiplicative group $\mathbf{F}_p^\times$ is cyclic of order $p-1$. If $p \neq 2$, then $p-1$ is even, so there is a unique surjective morphism of groups $\lambda_p : \mathbf{F}_p^\times \to \mathbf{Z}^\times$, where $\mathbf{Z}^\times$ is the multiplicative group consisting of 1 and -1.

In general, G being any group, a surjective morphism of groups $G \to \mathbf{Z}^\times$ is called a *quadratic character* of G. We have seen that for every *odd* prime p, there is a *unique* quadratic character of $\mathbf{F}_p^\times$.

For the even prime 2, we need to consider the *three* quadratic characters of the multiplicative group $(\mathbf{Z}/8\mathbf{Z})^\times$. The first one comes from the unique isomorphism of groups $\lambda_4 : (\mathbf{Z}/4\mathbf{Z})^\times \to \mathbf{Z}^\times$; indeed, the two groups in question have order 2, so they are isomorphic and there is only one isomorphism between them.

To define the second one, view $\mathbf{Z}^\times$ as a subgroup of $(\mathbf{Z}/8\mathbf{Z})^\times$ (consisting of $\bar{1}$ and $-\bar{1}$), so that the quotient group $(\mathbf{Z}/8\mathbf{Z})^\times/\mathbf{Z}^\times$ has order 2. There is thus a unique isomorphism of groups $\lambda_8 : (\mathbf{Z}/8\mathbf{Z})^\times/\mathbf{Z}^\times \to \mathbf{Z}^\times$, and it can be viewed as a quadratic character of $(\mathbf{Z}/8\mathbf{Z})^\times$.

The third quadratic character of $(\mathbf{Z}/8\mathbf{Z})^\times$ is simply the product $\lambda_4\lambda_8$, defined by $\lambda_4\lambda_8(x) = \lambda_4(x)\lambda_8(x)$ for every $x \in (\mathbf{Z}/8\mathbf{Z})^\times$. Out of these three quadratic characters, only λ_8 is *even* in the sense that $\lambda_8(-1) = 1$; the other two are *odd* because $\lambda_4(-1) = -1$ and $\lambda_4\lambda_8(-1) = -1$.

For every prime p, denote by $\mathbf{Z}_{(p)}$ the smallest subring of $\mathbf{Q}$ containing l^{-1} for every prime $l \neq p$. The morphism of rings $\mathbf{Z} \to \mathbf{F}_p$ can be extended uniquely to a morphism of rings $\mathbf{Z}_{(p)} \to \mathbf{F}_p$; its kernel is $p\mathbf{Z}_{(p)}$. We thus get a morphism of groups $\mathbf{Z}_{(p)}^\times \to \mathbf{F}_p^\times$ which is easily seen to be surjective. For $p \neq 2$, we may thus view λ_p as a quadratic character of $\mathbf{Z}_{(p)}^\times$.

Similarly, for $p = 2$, the morphism of rings $\mathbf{Z} \to \mathbf{Z}/8\mathbf{Z}$ can be extended uniquely to a morphism of rings $\mathbf{Z}_{(2)} \to \mathbf{Z}/8\mathbf{Z}$ (with kernel $8\mathbf{Z}_{(2)}$). We thus get a morphism of groups $\mathbf{Z}_{(2)}^\times \to (\mathbf{Z}/8\mathbf{Z})^\times$ which is easily seen to be surjective. We may thus view λ_4 and λ_8 as quadratic characters of $\mathbf{Z}_{(2)}^\times$.

Till now, we have only defined some quadratic characters. Here is our first observation: for every $a \in \mathbf{Z}_{(2)}^\times$,

$$\lambda_4(a) = (-1)^{\frac{a-1}{2}}, \qquad \lambda_8(a) = (-1)^{\frac{a^2-1}{8}}.$$

Without going into the proof, we clarify that these formulae have a meaning. Indeed, if $a \in \mathbf{Z}_{(2)}^\times$, then $a-1 \in 2\mathbf{Z}_{(2)}$, so $\frac{a-1}{2} \in \mathbf{Z}_{(2)}$. Now, $\mathbf{Z}^\times$ is a (multiplicatively written) vector space over $\mathbf{F}_2$ (of dimension 1), and hence a module over $\mathbf{Z}_{(2)}$, so the expression $(-1)^{\frac{a-1}{2}}$ has a meaning. Similarly, the expression $(-1)^{\frac{a^2-1}{8}}$ has a meaning for every $a \in \mathbf{Z}_{(2)}^\times$, for then $a^2-1 \in 8\mathbf{Z}_{(2)}$ and $\frac{a^2-1}{8} \in \mathbf{Z}_{(2)}$.

The foregoing formulae can be said to compute λ_4 and λ_8. Can we compute λ_p for odd primes p? In other words, is there a formula for $\lambda_p(a)$, valid for every $a \in \mathbf{Z}_{(p)}^\times$? Such a formula is precisely what the law of quadratic reciprocity gives.

Recall that for $p \neq 2$ the group $\mathbf{Z}_{(p)}^\times$ is collectively generated by -1, 2 and all odd primes $q \neq p$. Since λ_p is a morphism of groups, it is sufficient to give a formula for $\lambda_p(-1)$, $\lambda_p(2)$ and $\lambda_p(q)$.

The law of quadratic reciprocity asserts that for every prime $p \neq 2$,

$$\lambda_p(-1) = \lambda_4(p), \quad \lambda_p(2) = \lambda_8(p), \quad \lambda_p(q) = \lambda_q(\lambda_4(p)p),$$

for every odd prime $q \neq p$. It was first discovered by Euler and Legendre in their old age, and proved by the young Gauß. Since then, a number of different proofs have been given (many of them by Gauß himself), and it has been vastly generalised.

One of the simplest proofs of the law $\lambda_p(q) = \lambda_q(\lambda_4(p)p)$ is perhaps the one given by Rousseau in 1991. It consists in computing the product of all elements in the group $(\mathbf{F}_p^\times \times \mathbf{F}_q^\times)/\mathbf{Z}^\times$ in two different ways, by exploiting the isomorphism of groups $(\mathbf{Z}/pq\mathbf{Z})^\times \to \mathbf{F}_p^\times \times \mathbf{F}_q^\times$.

The law of quadratic reciprocity was generalised by Gauß, Jacobi, and Eisenstein to cubic and quartic reciprocity laws. For this purpose, they had to enlarge the field $\mathbf{Q}$ to $\mathbf{Q}(j)$ and $\mathbf{Q}(i)$ respectively, where j is a primitive third root of 1 ($j^3 = 1$, $j \neq 1$) and i is a primitive fourth root of 1 ($i^4 = 1$, $i^2 \neq 1$). Dirichlet found the analogue of quadratic reciprocity for the field $\mathbf{Q}(i)$. Eisenstein and Kummer made deep contributions to some cases of the l-tic reciprocity law in $\mathbf{Q}(\zeta)$, where ζ is a primitive l-th root of unity ($\zeta^l = 1$, $\zeta \neq 1$) and l is an odd prime.

But let us jump directly to Hilbert, who reformulated the quadratic reciprocity law as a *product formula* which made it possible to guess what the generalisation to m-tic reciprocity should be, for every $m > 1$ (over a number field which contains a primitive m-th root of unity).

The first notion we need is that of a *place* of $\mathbf{Q}$, which can be either finite or archimedean. A finite place of $\mathbf{Q}$ is just a prime number, and there is just one archimedean place, denoted ∞. Shortly we shall define the completion $\mathbf{Q}_v$ of $\mathbf{Q}$ at a place v. It will turn out that $\mathbf{Q}_\infty$ is just the field $\mathbf{R}$ of real numbers. For every finite place p of $\mathbf{Q}$, Hensel defined a new field called the field of p-adic numbers and denoted $\mathbf{Q}_p$. It is in terms of these fields that the mystery in the following definitions will be clarified.

For any two numbers $a, b \in \mathbf{Q}^\times$ and every place v of $\mathbf{Q}$, define $(a,b)_v \in \mathbf{Z}^\times$ by the following explicit but opaque rules.

If $v = \infty$, then $(a,b)_\infty = 1$ precisely when $a > 0$ or $b > 0$, so that $(a,b)_\infty = -1$ if $a < 0$ and $b < 0$. Here the definition is not so mysterious because $(a,b)_\infty = 1$ precisely when the equation $ax^2 + by^2 = 1$ has a solution $x, y \in \mathbf{R}$.

Now let v be a finite place of $\mathbf{Q}$, so that it is some prime number p. Note that every $x \in \mathbf{Q}^\times$ can be uniquely written as $x = p^{v_p(x)}u_x$, with $v_p(x) \in \mathbf{Z}$ and $u_x \in \mathbf{Z}_{(p)}^\times$. Let $a, b \in \mathbf{Q}^\times$, write

$$a = p^{v_p(a)}u_a, \quad b = p^{v_p(b)}u_b,$$

$$(v_p(a), v_p(b) \in \mathbf{Z}, u_a, u_b \in \mathbf{Z}_{(p)}^\times),$$

and put

$$t_{a,b} = (-1)^{v_p(a)v_p(b)} a^{v_p(b)} b^{-v_p(a)} = (-1)^{v_p(a)v_p(b)} u_a^{v_p(b)} u_b^{-v_p(a)},$$

which is visibly in $\mathbf{Z}_{(p)}^\times$. Define

$$(a,b)_p = \lambda_p(t_{a,b}) \;\; (p \neq 2), \quad (a,b)_2 = (-1)^{\frac{u_a-1}{2}\frac{u_b-1}{2}} \lambda_8(t_{a,b}).$$

As we have said, these definitions might seem unmotivated and contrived, but their real meaning will come out once we have defined the fields $\mathbf{Q}_p$.

We are ready to state Hilbert's product formula. It says that for $a, b \in \mathbf{Q}^\times$, we have $(a,b)_v = 1$ for almost all places v of $\mathbf{Q}$, and

$$\prod_v (a,b)_v = 1.$$

By unravelling the definitions, this product formula can be seen to be equivalent to the quadratic reciprocity law.

For example, when p and q are distinct odd (positive) primes, the definitions give $(p,q)_\infty = 1$,

$$(p,q)_2 = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}, \quad (p,q)_p = \lambda_p(q), \quad (p,q)_q = \lambda_q(p)$$

and $(p,q)_l = 1$ for every odd prime l different from p and q. So the product formula in this case becomes

$$(-1)^{\frac{p-1}{2}\frac{q-1}{2}} \lambda_p(q)\lambda_q(p) = 1,$$

which is equivalent to $\lambda_p(q) = \lambda_q(\lambda_4(p)p)$ in view of $\lambda_4(p) = (-1)^{\frac{p-1}{2}}$ and $\lambda_q(-1) = \lambda_4(q) = (-1)^{\frac{q-1}{2}}$.

The advantage of this reformulation of the quadratic reciprocity law as a product formula is that it is so neat, compact, and memorable. The disadvantage is that one does not quite understand where the symbol $(a,b)_p \in \mathbf{Z}^\times$ (for primes p) comes from. It can be properly understood only in terms of Hensel's p-adic numbers, to which we now turn.

It is best to first define the ring $\mathbf{Z}_p$ of p-adic integers. It is the "inverse limit" of the system of rings $\mathbf{Z}/p^n\mathbf{Z}$ and homomorphisms

$$\varphi_n : \mathbf{Z}/p^{n+1}\mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$$

(of reduction modulo p^n). Thus a p-adic integer $x \in \mathbf{Z}_p$ is a system of elements $x = (x_n)_{n>0}$ such that $x_n \in \mathbf{Z}/p^n\mathbf{Z}$ and $\varphi_n(x_{n+1}) = x_n$. Addition and multiplication are defined componentwise. It turns out that the ring $\mathbf{Z}_p$ is integral, and $\mathbf{Z}$ can be identified with a subring of $\mathbf{Z}_p$. The field $\mathbf{Q}_p$ is defined as the field of fractions of $\mathbf{Z}_p$.

The ring $\mathbf{Z}_p$ carries a natural topology, the coarsest topology for which all the projection morphisms $\mathbf{Z}_p \to \mathbf{Z}/p^n\mathbf{Z}$ are continuous. It induces a topology on $\mathbf{Q}_p$ for which it is locally compact and $\mathbf{Q}$ is a dense subset.

For $a, b \in \mathbf{Q}_p^\times$, one can define the symbol $(a, b)_p \in \mathbf{Z}^\times$ to be 1 if the equation $ax^2 + by^2 = 1$ has a solution $x, y \in \mathbf{Q}_p$, and $(a, b)_p = -1$ otherwise. One can check that for $a, b \in \mathbf{Q}^\times$, this new definition in terms of the solvability of $ax^2 + by^2 = 1$ in $\mathbf{Q}_p$ coincides with the previous explicit definition in terms of the quadratic characters λ_*.

There is an even better way of understanding the symbols $(a, b)_v \in \mathbf{Z}^\times$ (for v a place of $\mathbf{Q}$ and $a, b \in \mathbf{Q}_v^\times$). Let $M_v = \mathbf{Q}_v(\sqrt{\mathbf{Q}_v^\times})$ be the maximal Abelian extension of $\mathbf{Q}_v$ of exponent 2. It is a *minor miracle* that there is a unique isomorphism

$$r_v : \mathbf{Q}_v^\times / \mathbf{Q}_v^{\times 2} \to \mathrm{Gal}\,(M_v|\mathbf{Q}_v)$$

such that for every extension L of $\mathbf{Q}_v$ in M_v, the kernel of the composite map

$$\rho_L : \mathbf{Q}_v^\times \to \mathbf{Q}_v^\times / \mathbf{Q}_v^{\times 2} \to \mathrm{Gal}\,(M_v|\mathbf{Q}_v) \to \mathrm{Gal}\,(L|\mathbf{Q}_v)$$

is equal to the image of the norm map $L^\times \to \mathbf{Q}_v^\times$. For $a, b \in \mathbf{Q}_v^\times$, take $L = \mathbf{Q}_v(\sqrt{b})$; then $\rho_L(a)(\sqrt{b})$ is either $\sqrt{b}$ or $-\sqrt{b}$, and $(a, b)_v \in \mathbf{Z}^\times$ is precisely the sign such that

$$\rho_L(a)(\sqrt{b}) = (a, b)_v \sqrt{b} \qquad (L = \mathbf{Q}_v(\sqrt{b})).$$

More is true. Let K be any finite extension of $\mathbf{Q}_v$, let $m > 0$ be any integer, and let M be the maximal Abelian extension of K of exponent dividing m. If K happens to contain a primitive m-th root of unity, then $M = K(\sqrt[m]{K^\times})$, by Kummer theory. It is a *minor miracle* that there is a unique isomorphism

$$r_{m,K} : K^\times / K^{\times m} \to \mathrm{Gal}\,(M|K)$$

such that for every extension L of K in M, the kernel of the composite map

$$\rho_L : K^\times \to K^\times / K^{\times m} \to \mathrm{Gal}\,(M|K) \to \mathrm{Gal}\,(L|K)$$

is equal to the image of the norm map $N_{L|K} : L^\times \to K^\times$. [At the finite places, there is an additional requirement which we have omitted because we have not defined the relevant concepts. In short, let $M_0 \subset M$ denote the maximal unramified extension of K in M; the groups $K^\times / N_{M_0|K}(M_0^\times)$ and $\mathrm{Gal}\,(M_0|K)$ are both cyclic (of order m) with a canonical generator, and the requirement is that ρ_{M_0} send the generator of $K^\times / N_{M_0|K}(M_0^\times)$ to the generator of $\mathrm{Gal}\,(M_0|K)$. This requirement is automatic if $m = 2$.]

With this minor miracle in hand, one could go on to discuss the general reciprocity law, but let us stick to the classical case where the presence of a primitive m-th root of unity is required.

Suppose therefore that K contains a primitive m-th root of unity. For $a, b \in K^\times$, take $L = K(\sqrt[m]{b})$; then $\rho_L(a)(\sqrt[m]{b})$ and $\sqrt[m]{b}$ differ by an m-th root of unity in μ_m, and one can define $(a, b)_{m,K} \in \mu_m$ by the requirement that

$$\rho_L(a)(\sqrt[m]{b}) = (a, b)_{m,K} \sqrt[m]{b} \qquad (L = K(\sqrt[m]{b})).$$

This is a generalisation of the previous case $m = 2$, $K = \mathbf{Q}_v$, where we denoted $(a, b)_{m,K}$ simply by $(a, b)_v$. This is the local ingredient we need in order to state the classical reciprocity laws.

Let us now turn to a finite extension F of $\mathbf{Q}$ (also called a number field) and explain what is meant by a *place* of F. As in the case of $\mathbf{Q}$ above, places come in two varieties: finite and archimedean. To a finite place v corresponds a prime number p, and v is said to be a p-adic place. An archimedean place can be real or imaginary.

A real place of F is simply an embedding $F \to \mathbf{R}$. An imaginary place of F is an embedding $\iota : F \to \mathbf{C}$ such that $\iota(F) \not\subset \mathbf{R}$, except that two embeddings ι_1, ι_2 determine the same imaginary place if they differ by the conjugation $z \mapsto \bar{z}$ $(i \mapsto -i)$ in $\mathbf{C}$: if $\iota_1(a) = \overline{\iota_2(a)}$ for every $a \in F$. We see that an archimedean place of F is really an equivalence class of embeddings $F \to \mathbf{C}$, two embeddings being equivalent if they differ by an element of $\mathrm{Gal}\,(\mathbf{C}|\mathbf{R})$. Every F has at least one and at most finitely many archimedean places.

Similarly, for every prime number p, a p-adic place of F is an equivalence class of embeddings $F \to \bar{\mathbf{Q}}_p$, where two embeddings are equivalent if they differ by an element of $\mathrm{Gal}\,(\bar{\mathbf{Q}}_p|\mathbf{Q}_p)$. Here, $\bar{\mathbf{Q}}_p$ is a fixed algebraic closure of $\mathbf{Q}_p$. Every F has at least one and at most finitely many p-adic places (for every prime p).

Recall that the field $\mathbf{Q}_p$ (p prime) carries a natural topology which makes it a locally compact field. As a result, every algebraic closure of $\mathbf{Q}_p$ also carries a natural topology (but $\bar{\mathbf{Q}}_p$ is not locally compact). Also, the group $\mathrm{Gal}\,(\bar{\mathbf{Q}}_p|\mathbf{Q}_p)$ is far more complicated than $\mathrm{Gal}\,(\mathbf{C}|\mathbf{R})$.

The next notion we need is that of the completion F_v of F at a place v. If v is real, then $F_v = \mathbf{R}$. If v is imaginary, then $F_v = \mathbf{C}$. For a p-adic place v of F, the completion F_v is defined to be the closure of $\iota(F)$ in $\bar{\mathbf{Q}}_p$, where $\iota : F \to \bar{\mathbf{Q}}_p$ is an embedding representing the place v. It is a finite extension of $\mathbf{Q}_p$, uniquely determined by

F and v, and $[F : \mathbf{Q}] = \sum_{v|p}[F_v : \mathbf{Q}_p]$, just as $[F : \mathbf{Q}] = \sum_{v|\infty}[F_v : \mathbf{R}]$, where $v|p$ means that v is a p-adic place and $v|\infty$ means that v is an archimedean place.

Till now the number field F has been arbitrary. We have defined the notion of a place v of F, and the completion F_v of F at v. Now let $m > 0$ be an integer, and suppose that F contains a primitive m-th root of unity. [If $m > 2$, then F does not have any real places.] Clearly, every completion F_v also contains a primitive m-th root of unity, making it possible to define $(a, b)_{m,F_v} \in \mu_m$ for $a, b \in F_v^\times$, as we have seen. [At an imaginary place v, we take $(a, b)_{m,\mathbf{C}} = 1$ for all $a, b \in \mathbf{C}^\times$ because $\mathbf{C}$ is algebraically closed. For this reason, imaginary places play no role in what follows.]

The main theorem, which encompasses all classical reciprocity laws, states that if we start with $a, b \in F^\times$, then $(a, b)_{m,F_v} = 1$ for almost all v, and the product formula

$$\prod_v (a, b)_{m,F_v} = 1$$

holds (in the group μ_m). Quadratic reciprocity is the special case $F = \mathbf{Q}$, $m = 2$. Cubic reciprocity (which we have not recalled) is the special case $F = \mathbf{Q}(j)$, $m = 3$. Quartic reciprocity is the special case $F = \mathbf{Q}(i)$, $m = 4$. Dirichlet's analogue of quadratic reciprocity is the special case $F = \mathbf{Q}(i)$, $m = 2$. Eisenstein, Kummer and Takagi's work on l-tic reciprocity (for an odd prime l) is the special case $F = \mathbf{Q}(\zeta)$ (where $\zeta^l = 1$, $\zeta \neq 1$) and $m = l$.

When F is an arbitrary number field and $m = 2$, we get the quadratic reciprocity law in F, due to Hilbert. We have $(a, b)_{2,F_v} = 1$ if and only if the equation $ax^2 + by^2 = 1$ has a solution $x, y \in F_v$; otherwise $(a, b)_{2,F_v} = -1$, just as in the special case $F = \mathbf{Q}$.

It is difficult to appreciate just how much information is packed into this single neat product formula. To unravel this information in the case of some particular number field F (containing a primitive m-th root of unity), we need to determine the places of F, and more importantly to give an explicit formula for $(a, b)_{m,F_v}$ at every place v. This quest has given rise to some of the deepest and most sublime mathematics ever dreamt of by a human mind.

The only shortcoming of the above product formula is that it is applicable only to those number fields which contain a primitive m-th root of unity. This restriction has been removed by Takagi, Artin and Hasse, who came up with the general reciprocity law. I hope to discuss it on some future occasion and show how Chevalley's invention of idèles provides a conceptual understanding of the general law, just as Hensel's invention of p-adic numbers provides a conceptual understanding of the classical laws.

Let us end with the provenance of the word *reciprocity*. It was first used by Legendre to reflect the fact that when p and q are distinct odd primes and one of them is $\equiv 1 \bmod 4$, then $\lambda_p(q) = \lambda_q(p)$, which is sometimes written more simply as $(q/p) = (p/q)$. In words: the value of λ_p at q is the same as the value of λ_q at p, or q is a square modulo p if and only if p is a square modulo q. The meaning of the word got reinforced with similar formulae such as $(a/b) = (b/a)$ which express special cases of other classical reciprocity laws. Since then, the notion of reciprocity has become a central tenet of Arithmetic, largely thanks to Robert Langlands, as attested by Roger Godement:

Legendre a deviné la formule et Gauss est devenu instatanément célèbre en la prouvant. En trouver des généralisations, par exemple aux anneaux d'entiers algébriques, ou d'autres démonstrations a constitué un sport national pour la dynastie allemande suscité par Gauss jusqu'à ce que le reste du monde, à commencer par le Japonais Takagi en 1920 et à continuer par Chevalley une dizaine d'années plus tard, découvre le sujet et, après 1945, le fasse exploser. Gouverné par un Haut Commissariat qui surveille rigoureusement l'alignement de ses Grandes Pyramides, c'est aujourd'hui l'un des domaines les plus respectés des Mathématiques. [5, p. 313].

The paper which has most influenced my point of view is [10]. The reader who wants to see the first part of this Note worked out in every detail can consult my *Eight lectures on quadratic reciprocity* [4].

The study of reciprocity laws led to class field theory. There is a fairly large number of books on this subject, starting with Hasse [6], Chevalley [3] and Artin-Tate [1]. A comprehensive account can be found in [2]. Other sources include the books by Weil [11], Serre [9], and Neukirch [8], and the online notes of Milne [7].

Acknowledgments

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