Topology of Puzzle Rings

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1. Introduction to Topology and Knot Theory

Here we treat plane figures and solid figures as topological spaces. A *plane figure* is a subset of the 2-dimensional Euclidean space $\mathbb{R}^2$ and a *solid figure* is a subset of the 3-dimensional Euclidean space $\mathbb{R}^3$. In general, a *figure* is a subset of the n-dimensional Euclidean space $\mathbb{R}^n$ for some natural number n.

Two figures X and Y are *homeomorphic* if there exists a continuous bijection $f : X \to Y$ such that the inverse map $f^{-1} : Y \to X$ is also continuous. Then we denote it by $X \cong Y$. Such a map f is said to be a *homeomorphism* from X to Y.

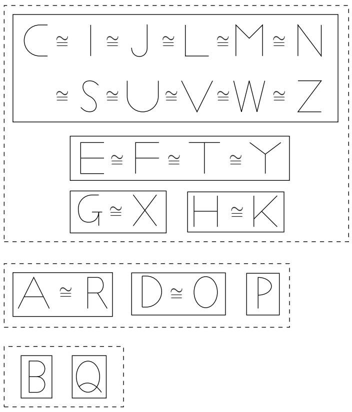

Fig. 1.1

For example, we consider the 26 alphabet capital letters as plane figures. Here we think that each of them is a finite union of line segments and curves. Namely they are 1-dimensional and have no areas. Then they are classified up to homeomorphism as illustrated in Fig. 1.1. In Fig. 1.1, a real line rectangle describes a homeomorphism class and a dotted line rectangle describes a *homotopy equivalence* class. Here we omit the definition of homotopy equivalence. But we note here that the homotopy equivalence classification of these 26 letters is in one to one correspondence with the homeomorphism classification of 26 boldface alphabet capital letters as illustrated in Fig. 1.2. Here each boldface letter is a regular neighbourhood of the corresponding letter and is

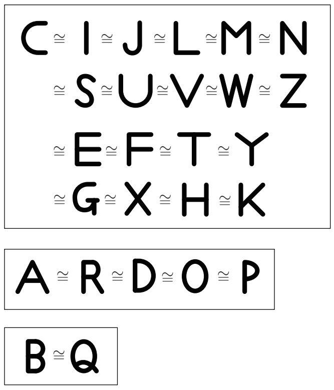

Fig. 1.2

a compact planar surface with boundary. They are completely classified by the Euler characteristic or the first Betti number.

For figures in the same n-dimensional Euclidean space $\mathbb{R}^n$, there is an equivalence relation that is stronger than the homeomorphism. Let X and Y be subsets of $\mathbb{R}^n$. We say that X and Y are *ambient isotopic* if there exists an orientation preserving homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(X) = Y$. Then we denote it by $X \approx Y$.

Two mutually homeomorphic plane figures that are not mutually ambient isotopic in $\mathbb{R}^2$ are illustrated in Fig. 1.3. Note that if we think them as solid figures, then they are ambient isotopic in $\mathbb{R}^3$.

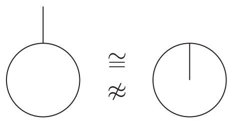

Fig. 1.3

A *knot* is a simple closed curve in $\mathbb{R}^3$. Knot theory studies whether or not two given knots are ambient isotopic in $\mathbb{R}^3$. By definition, any two knots are mutually homeomorphic. Two mutually non-ambient isotopic knots, 0_1 and 3_1, are illustrated in Fig. 1.4.

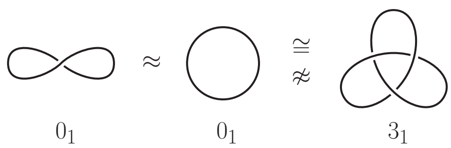

Fig. 1.4

2. Topology of Puzzle Rings

There are many studies on puzzle rings. However it seems to me that not so many studies on them from topological viewpoint are done yet. Here we consider a puzzle ring with *hard part* and *soft part*. A hard part is rigid and made of metal for example. A soft part is pliable and made of string for example. This can be formulated as follows. Let $X_i = H_i \cup S_i$ be a subset of $\mathbb{R}^3$ with $H_i \cap S_i = \emptyset$ for $i = 1, 2$. We say that X_1 and X_2 are *equivalent* if there exist an orientation preserving isometry $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(H_1) = H_2$ and an orientation preserving homeomorphism $g : \mathbb{R}^3 \to \mathbb{R}^3$ that is pointwisely fixed on H_2 such that $g(f(S_1)) = S_2$. Below we consider the case that $X = H \cup S$ is a finite graph embedded in $\mathbb{R}^3$. Then we can apply spatial graph theory to puzzle ring problem. Now we consider the following problem. From now on we do not stick to using only mathematically defined terminologies.

Problem 1. In Fig. 2.1, remove the soft part from the hard part.

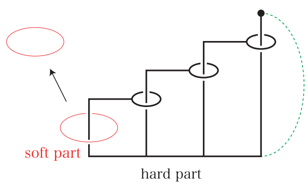

Fig. 2.1

To be more mathematical, we reformulate the problem as follows. The dotted line in Fig. 2.1 is an imaginary line.

Problem 2. How many times does the soft part need to go across the dotted line to be away from the hard part as illustrated in Fig. 2.1?

Note that Problem 2 can be graded as illustrated in Fig. 2.2.

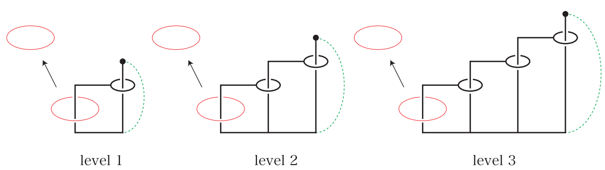

Fig. 2.2

The answer for level n is 2^n. In particular the answer for Problem 2 is $2^3 = 8$. An actual deformation that shows 8 is sufficient is illustrated in Fig. 2.3.

However it will be unclear that the answer is 2^n in general. Here we think the situation fully topologically. Namely we suppose that the hard part is also soft. Then we have a deformation as illustrated in Fig. 2.4. It is clear by the final illustration in Fig. 2.4 that the soft part bounds a disk that intersects the dotted line transversally at 8 points. This fact is common for all illustrations in Fig. 2.4 if we allow the disk to be topological. Then we can shrink the soft part along the topological disk. Then the soft part will become sufficiently small and away from the hard part after going across the dotted line 8 times.

It will be easy to image the solution for level n from this solution for $n = 3$.

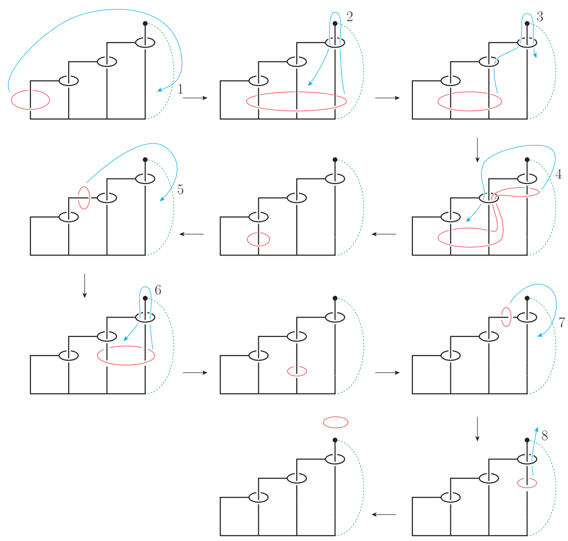

Fig. 2.3

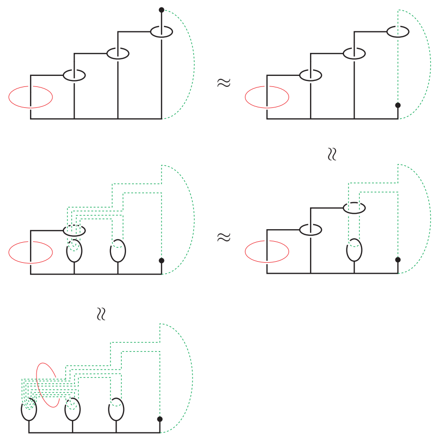

Fig. 2.4

It is necessary to show that 2^n is necessary. It is shown in [1] by group theoretic argument.

Recently the author found a geometric proof using covering space theory. It may be essentially the same but at least for the author it is very understandable. It will appear in [2] together with certain generalisations.

A way to make a puzzle ring with hard part and soft part is illustrated in Fig. 2.5. In such a way we can produce a variety of puzzle rings

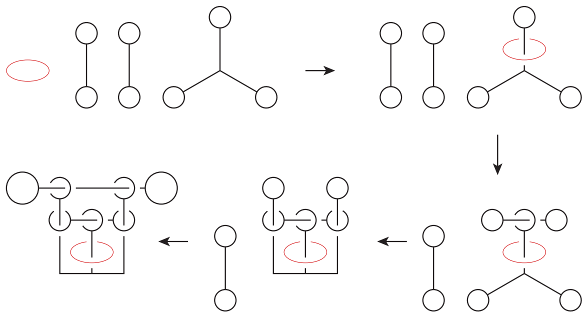

Fig. 2.5

with hard part and soft part as illustrated in Fig. 2.6.

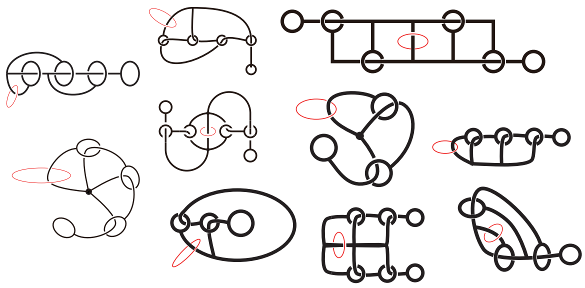

Fig. 2.6

References

[1] J. Przytycki and A. Sikora, Topological insights from the Chinese rings, *Proc. Amer. Math. Soc.* **130**(3) (2002) 893–902.

[2] K. Taniyama, Site-specific Gordian distances of spatial graphs, in preparation.

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