Ramsey's Theorem, Reverse Mathematics and Nonstandard Models

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1. Ramsey's Theorem for Pairs and Weaker Principles

In layman's term, Ramsey's Theorem for pairs can be stated as follows: In a party of infinitely many participants, we can always find an infinite group of people such that all of them knew each other or all of them are mutual strangers. The precise statement of the general version of Ramsey's Theorem says:

Theorem 1.1 (Ramsey [12]). *Let* $\mathbb{N}$ *denote the set of natural numbers. Any* $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$ *has an infinite* homogeneous set $H \subseteq \mathbb{N}$, *namely,* f *is constant on* $[H]^n$; *here* $[X]^n$ *stands for the set of all* n*-element subsets of* X.

If we think of f as a k-colouring of the n-element subsets of natural numbers, then there is an infinite set H whose n-element subsets have the same colour. It is customary to think of statements of the form "for all f there exists H..." as a game: "the opponent poses a problem (e.g. a colouring scheme) and we are to provide a solution (e.g. the infinite homogeneous set)".

The version above is denoted by RT^n_k. Our main focus is on RT^2_2 — Ramsey's Theorem for Pairs. In the example given above, we "colour" a pair of people "blue" if they knew each other, and colour them "red" otherwise. Then the homogeneous set H is the group that with the desired property. We now give a proof of RT^2_2. Let f be a colouring of pairs, say by red and blue. We first find an infinite subset C of natural numbers on which f is "stable", i.e. for all $x \in C$, the limit $\lim_{y \in C} f(x, y)$ exists. We call such a set C *cohesive for* f. Next we consider the following two sets: $D^R = \{x \in C : x \text{ is "eventually red"}\}$ and $D^B = \{x \in C : x \text{ is "eventually blue"}\}$. One of them must be infinite, say it is D^R. Now it is fairly easy to select the elements of a red homogeneous set from D^R: Let a_0 be the least element in D^R. Suppose that we have selected $a_0 < a_1 < \cdots < a_k$, let a_{k+1} be the first element larger than a_k such that (a_i, a_{k+1}) is coloured red for all $i \leq k$. The existence of a_{k+1} is guaranteed by the stability.

We extract two combinatorial principles out of the proof: Let R be an infinite subset of natural numbers and $R_s = \{t | \langle s, t\rangle \text{ is in } R\}$ where $\langle s, t\rangle$ stands for the Gödel coding of pairs. A set G is said to be R-cohesive if for all s, either $G \cap R_s$ is finite or $G \cap (\mathbb{N} \setminus R_s)$ is finite. The cohesive principle COH states that for every R, there is an infinite G that is R-cohesive. The other principle is called the stable Ramsey's Theorem for pairs, denoted by SRT^2_2 which states that every stable colouring of pairs has a solution. The principles COH and SRT^2_2 were studied by Cholak, Jockusch and Slaman [1], where they showed

Theorem 1.2 (Cholak, Jockusch and Slaman).

$$\mathsf{RT}^2_2 = \mathsf{SRT}^2_2 + \mathsf{COH}.$$

There are many other principles which are corollaries of Ramsey's Theorem for pairs. For instance, the principle ADS of ascending or descending sequence states that every infinite linearly ordered set contains an infinite subsequence that is either increasing or decreasing. The Chain and Antichain Principle CAC states that every infinite partially ordered set has an infinite chain or antichain.

2. Introducing Reverse Mathematics

One of the main objectives of reverse mathematics is to study the relative strength of mathematical theorems. In this case, we are interested in the relative strength of combinatorial principles, in particular the principles related to Ramsey's theorem. It turns out that the most interesting ones are those implied by Ramsey's theorem for pairs. For example, it is natural to ask whether COH or SRT^2_2 is as strong as RT^2_2, and whether ADS implies RT^2_2. To put it more generally, what are the logical consequences and what is the strength of a combinatorial principle, for example, Ramsey's Theorem?

To answer these questions, one needs to bring in tools from logic. For example, the answer may depend on analysing the complexity of the homogeneous set H. Also, one needs logic to determine if one principle P implies, or does not imply, the principle Q. It is usually more challenging to show that P does not imply Q. As we know from logic that one way to demonstrate that $P \nRightarrow Q$ is to "make" P true and Q false. However, since these combinatorial principles are all true theorems in mathematics (provable within the standard system of set theory), how can one make it false?

Thus we have to work in some weaker axiomatic system Γ, and demonstrate that "Γ proves P but not Q". Usually, we will have a hierarchy of axiomatic systems $\Gamma_0 < \Gamma_1 < \ldots$ as our benchmarks and their relative strength has been established in a strictly increasing order so that Γ_i is strictly weaker than Γ_j for $i < j$ (as indicated by the "<" symbol between the systems). Therefore, to show that P does not prove Q, it suffices to show that Γ_i proves P and on the other hand Q proves Γ_j for some $j > i$. Notice that the last step requires that we prove axiom Γ_j from a theorem Q, which reverses the usual mathematical practice of proving theorems from axioms. This is why the term "reverse mathematics" is coined. The standard reference book for reverse mathematics is Simpson [14].

We now introduce two most commonly used axiom systems in the study of reverse mathematics, namely the subsystems of first- and second-order arithmetic. Recall that the language of first order Peano Arithmetic consists of a constant symbol 0, three function symbols $S, +, \times$ (where $S(x) = x + 1$ for any number x) and a binary predicate $<$. Formulas over the language of arithmetic naturally form a hierarchy by the number of alternating blocks of quantifiers, giving us the usual arithmetic hierarchy. Formulas with n alternating blocks of quantifiers with the leading one existential (resp. universal) are called Σ^0_n (resp. Π^0_n). The superscript 0 indicates that the formulas are first-order. For example, $(\forall x, y, z > 2)[x^{2015} + y^{2015} \neq z^{2015}]$ is Π^0_1 (here x^{2015} is a shorthand for the product of 2015 copies of x); and the twin prime conjecture is Π^0_2. Also, given an axiom system, the Δ^0_n formulas are those having two equivalent forms, one Σ^0_n and Π^0_n, provable within the system.

Let $I\Sigma^0_n$ denote the mathematical induction schema for Σ^0_n-formulas, and $B\Sigma^0_n$ denote the Bounding Principle for Σ^0_n-formulas. $B\Sigma^0_n$ says that every Σ^0_n definable function maps a finite set onto a finite set. By a theorem of Kirby and Paris [9]

$$\cdots \Rightarrow I\Sigma^0_{n+1} \Rightarrow B\Sigma^0_{n+1} \Rightarrow I\Sigma^0_n \Rightarrow \ldots$$

This gives us a set of benchmarks in first-order arithmetic.

The other set of benchmarks is based on the collection of subsystems of second-order arithmetic which is used in reverse mathematics. In second-order arithmetic, the variables and quantifiers in a sentence can range over sets or relations. For example, "every nonempty subset has a least element" can be written as

$$(\forall X)((\exists x)(x \in X) \to (\exists x)(\forall y)(x \in X \,\&\, y \in X \to y \geq x)).$$

Complexity can be defined similarly. For example, the above sentence is Π^1_1. Here we only list three of the subsystems which are needed in the sequel: RCA_0 which contains Σ^0_1-induction and Δ^0_1-comprehension: For any Δ^0_1-formula φ, $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$; WKL_0 which is RCA_0 plus weak König Lemma saying that every infinite binary tree has an infinite path; and ACA_0 which is RCA_0 plus arithmetical comprehension. Their relative strengths are known:

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0.$$

We also need the notion of models. A model $\mathcal{M}$ of second-order arithmetic is a mathematical structure $(M, 0, S, +, \times, <, X)$ where $(M, 0, S, +, \times, <)$ is its first-order part and $X \subseteq 2^M$ is the second-order part. The set variables are interpreted as members of X. For example, if $\mathcal{M}$ is a model of RCA_0, then its second-order part X is closed under Turing reducibility and Turing join. In short, we have two measures of strength: the first-order measure which depends on the strength of induction satisfied and the second-order measure which is provided by the richness of set existence. With the concept of hierarchies available, we can recast the motivating questions as follow:

(1) Suppose the colouring function f is recursive. What is the minimal syntactical complexity of a solution?
(2) To which system in reverse mathematics does Ramsey's Theorem correspond? For example, does RT^2_2 imply ACA_0?

(3) What are the first-order consequences of Ramsey's Theorem? For example does RT^2_2 imply $I\Sigma^0_n$ for $n > 1$?
(4) Does SRT^2_2 imply RT^2_2? More precisely, if a model of RCA_0 whose second-order part X contains solutions for all stable colourings in X, must X contain solutions for all colourings in X?

3. Earlier Results

We now give a list of key results in the development of the subject. Some of the early studies were motivated by effective mathematics, and presented here in the language of reverse mathematics.

Theorem 3.1 (Jockusch [8]). *Over* RCA_0,

$$\mathsf{ACA}_0 \Leftrightarrow \mathsf{RT}^3_2 \Leftrightarrow \mathsf{RT}^n_k$$

where $n, k \geq 3$ *and*

$$\mathsf{ACA}_0 \Rightarrow \mathsf{RT}^2_2 \quad \textit{and} \quad \mathsf{WKL}_0 \nRightarrow \mathsf{RT}^2_2.$$

Theorem 3.2 (Hirst [7]). *Over* RCA_0,

$$\mathsf{SRT}^2_2 \Rightarrow B\Sigma_2.$$

This sets a lower bound for the first-order strength of SRT^2_2 and hence RT^2_2.

Theorem 3.3 (Seetapun and Slaman [13]). *Over* RCA_0,

$$\mathsf{RT}^2_2 \nRightarrow \mathsf{ACA}_0.$$

Seetapun's proof made clever use of trees, which leads to the Seetapun Conjecture: $\mathsf{RT}^2_2 \Rightarrow \mathsf{WKL}_0$.

To determine an upper bound of first-order strength, conservation results are often used. One of the earliest conservation results was due to Harrington, who showed that WKL_0 is Π^1_1-conservative over RCA_0, i.e. any Π^1_1-statement that is provable in WKL_0 is already provable in RCA_0.

Theorem 3.4 (Cholak, Jockusch and Slaman [1]). RT^2_2 *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + I\Sigma_2$.

Corollary 3.5 (Cholak, Jockusch and Slaman [1]). *Over* RCA_0,

$$\mathsf{RT}^2_2 \nRightarrow I\Sigma_3.$$

4. Recent Results

Since the work of Cholak, Jockusch and Slaman, the exact strength of RT^2_2 became the central problem of study and a major focus of attention in reverse mathematics, with many attempts made at solving it. The extensive study created a new paradigm for the field. For example, the collection of subsystems (called the *big five*) widely used before the turn of the century for benchmarking the strength of a mathematical theorem was no longer sufficient. In fact, for combinatorial principles related to RT^2_2, a linear ordering of measures to classify the strength of a system will not work. Today the picture looks rather like a "zoo". Hirschfeldt and Shore [6] made further progress on the exact strength of many important combinatorial principles weaker than RT^2_2. For instance, they showed that ADS is strictly weaker than RT^2_2. However, three major questions remained open: (1) Seetapun's Conjecture; (2) Over RCA_0, does SRT^2_2 imply RT^2_2? (3) Does SRT^2_2 or RT^2_2 imply $I\Sigma_2$?

The first problem was solved by Jiayi Liu [10] who showed that

Theorem 4.1 (Jiayi Liu [10]). *Over* RCA_0,

$$\mathsf{RT}^2_2 \nRightarrow \mathsf{WKL}_0.$$

However, the solution for (2) and (3) remained elusive. The most natural approach was to show that stable colourings always had a low solution (here the word "low" is a technical term in recursion theory). Or equivalently, every Δ^0_2-set contains or is disjoint from an infinite low set. However, Downey, Hirschfeldt, Lempp and Solomon [5] showed that there is a Δ^0_2 set D such that neither D nor $\mathbb{N} \setminus D$ contains an infinite low subset, blocking the seemingly promising approach.

It was here that the method of nonstandard models of arithmetic came into play. We had been working on recursion theory on nonstandard models of arithmetic for more than ten years at the time. Chitat Chong [2] suggested that one should perhaps look at nonstandard models of fragments of arithmetic, because the theorem of Downey, Hirschfeldt, Lempp and Solomon relied heavily on the inductive strength of the standard model of arithmetic, with a proof involving infinite injury construction that required $I\Sigma_2$. On the other hand, in nonstandard models things behave differently. For example, there is a model of $B\Sigma_2$ and not $I\Sigma_2$, first constructed by Mytilinaios and Slaman [11], in which every incomplete Δ^0_2 set is low. In collaboration with Theodore A. Slaman, after almost ten years work this approach has turned out to be fruitful:

Theorem 4.2 (Chong, Slaman and Yang [3]). *Over* RCA_0,

$$\mathsf{SRT}^2_2 \not\Rightarrow \mathsf{RT}^2_2$$

$$\mathsf{SRT}^2_2 \not\Rightarrow I\Sigma_2.$$

Theorem 4.3 (Chong, Slaman and Yang [4]).

$$\mathsf{RT}^2_2 \not\Rightarrow I\Sigma_2.$$

We end this survey with the following open questions: Does SRT^2_2 imply RT^2_2 in an ω-model, i.e. a model with first-order part the set of natural numbers? What is the proof-theoretic ordinal of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$?

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