Ramsey’s Theorem, Reverse Mathematics and Nonstandard Models

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1. Ramsey’s Theorem for Pairs and Weaker Principles

In layman’s term, Ramsey’s Theorem for pairs can be stated as follows: In a party of infinitely many participants, we can always find an infinite group of people such that all of them knew each other or all of them are mutual strangers. The precise statement of the general version of Ramsey’s Theorem says:

**Theorem 1.1 (Ramsey [12]).** Let \( \mathbb{N} \) denote the set of natural numbers. Any \( f : [\mathbb{N}]^k \to \{0, 1, \ldots, k - 1\} \) has an infinite homogeneous set \( H \subseteq \mathbb{N} \), namely, \( f \) is constant on \( [H]^k \); here \( [X]^n \) stands for the set of all \( n \)-element subsets of \( X \).

If we think of \( f \) as a \( k \)-colouring of the \( n \)-element subsets of natural numbers, then there is an infinite set \( H \) whose \( n \)-element subsets have the same colour. It is customary to think of statements of the form “for all \( f \) there exists \( H \)” as a game: “the opponent poses a problem (e.g. a colouring scheme) and we are to provide a solution (e.g. the infinite homogeneous set)”. The version above is denoted by \( \text{RT}^k_2 \). Our main focus is on \( \text{RT}^2_2 \) — Ramsey’s Theorem for Pairs. In the example given above, we “colour” a pair of people “blue” if they knew each other, and colour them “red” otherwise. Then the homogeneous set \( H \) is the group that with the desired property. We now give a proof of \( \text{RT}^2_2 \). Let \( f \) be a colouring of pairs, say by red and blue. We first find an infinite subset \( C \) of natural numbers on which \( f \) is “stable”, i.e. for all \( x \in C \), the limit \( \lim_{y \to x} f(x, y) \) exists. We call such a set \( C \) cohesive for \( f \). Next we consider the following two sets: \( D^R = \{ x \in C : x \text{ is “eventually red”}\} \) and \( D^B = \{ x \in C : x \text{ is “eventually blue”}\} \). One of them must be infinite, say it is \( D^R \). Now it is fairly easy to select the elements of a red homogeneous set from \( D^R \): Let \( a_0 \) be the least element in \( D^R \). Suppose that we have selected \( a_0 < a_1 < \cdots < a_k \), let \( a_{k+1} \) be the first element larger than \( a_k \) such that \((a_i, a_{i+1})\) is coloured red for all \( i \leq k \). The existence of \( a_{k+1} \) is guaranteed by the stability.

We extract two combinatorial principles out of the proof: Let \( R \) be an infinite subset of natural numbers and \( R_s = \{ t | (s, t) \text{ is in } R \} \) where \( (s, t) \) stands for the Gödel coding of pairs. A set \( G \) is said to be \( R \)-cohesive if for all \( s \), either \( G \cap R_s \) is finite or \( G \cap (\mathbb{N} \setminus R_s) \) is finite. The cohesive principle \( \text{COH} \) states that for every \( R \), there is an infinite \( G \) that is \( R \)-cohesive. The other principle is called the stable Ramsey’s Theorem for pairs, denoted by \( \text{SRT}^2_2 \) which states that every stable colouring of pairs has a solution. The principles \( \text{COH} \) and \( \text{SRT}^2_2 \) were studied by Cholak, Jockusch and Slaman [1], where they showed

**Theorem 1.2 (Cholak, Jockusch and Slaman).**

\[ \text{RT}^2_2 = \text{SRT}^2_2 + \text{COH}. \]

There are many other principles which are corollaries of Ramsey’s Theorem for pairs. For instance, the principle \( \text{ADS} \) of ascending or descending sequence states that every infinite linearly ordered set contains an infinite subsequence that is either increasing or decreasing. The Chain and Antichain Principle \( \text{CAC} \) states that every infinite partially ordered set has an infinite chain or antichain.

2. Introducing Reverse Mathematics

One of the main objectives of reverse mathematics is to study the relative strength of mathematical theorems. In this case, we are interested in the relative strength of combinatorial principles, in particular the principles related to Ramsey’s theorem. It turns out that the most interesting ones are those implied by Ramsey’s theorem for pairs. For example, it is natural to ask whether \( \text{COH} \) or \( \text{SRT}^2_2 \) is as strong as \( \text{RT}^2_2 \), and whether \( \text{ADS} \) implies \( \text{RT}^2_2 \). To put it more generally, what are the logical consequences and what is the strength of
a combinatorial principle, for example, Ramsey’s Theorem?

To answer these questions, one needs to bring in tools from logic. For example, the answer may depend on analysing the complexity of the homogeneous set \( H \). Also, one needs logic to determine if one principle \( P \) implies, or does not imply, the principle \( Q \). It is usually more challenging to show that \( P \) does not imply \( Q \). As we know from logic that one way to demonstrate that \( P \Rightarrow Q \) is to “make” \( P \) true and \( Q \) false. However, since these combinatorial principles are all true theorems in mathematics (provable within the standard system of set theory), how can one make it false?

Thus we have to work in some weaker axiomatic system \( \Gamma \), and demonstrate that “\( \Gamma \) proves \( P \) but not \( Q \)”. Usually, we will have a hierarchy of axiomatic systems \( \Gamma_0 < \Gamma_1 < \ldots \) as our benchmarks and their relative strength has been established in a strictly increasing order so that \( \Gamma_i \) is strictly weaker than \( \Gamma_j \) for \( i < j \) (as indicated by the “\( < \)” symbol between the systems). Therefore, to show that \( P \) does not prove \( Q \), it suffices to show that \( \Gamma_i \) proves \( P \) and on the other hand \( Q \) proves \( \Gamma_j \) for some \( j > i \). Notice that the last step requires that we prove axiom \( \Gamma_i \) from a theorem \( Q \), which reverses the usual mathematical practice of proving theorems from axioms. This is why the term “reverse mathematics” is coined. The standard reference book for reverse mathematics is Simpson [14].

We now introduce two most commonly used axiom systems in the study of reverse mathematics, namely the subsystems of first- and second-order arithmetic. Recall that the language of first order Peano Arithmetic consists of a constant symbol 0, three function symbols \( S, +, \times \) (where \( S(x) = x + 1 \) for any number \( x \)) and a binary predicate \( < \). Formulas over the language of arithmetic naturally form a hierarchy by the number of alternating blocks of quantifiers, giving us the usual arithmetic hierarchy. Formulas with \( n \) alternating blocks of quantifiers with the leading one existential (resp. universal) are called \( \Sigma_n \) (resp. \( \Pi_n \)). The superscript 0 indicates that the formulas are first-order. For example, \((\forall x, y, z > 2)(x^{2015} + y^{2015} \neq z^{2015})\) is \( \Pi_1 \) (here \( x^{2015} \) is a shorthand for the product of 2015 copies of \( x \)), and the twin prime conjecture is \( \Pi_0 \). Also, given an axiom system, the \( \Delta_0 \) formulas are those having two equivalent forms, one \( \Sigma_n \) and \( \Pi_m \), provable within the system.

Let \( \Sigma^0_n \) denote the mathematical induction schema for \( \Sigma^0_n \)-formulas, and \( B\Sigma^0_n \) denote the Bounding Principle for \( \Sigma^0_n \)-formulas. \( B\Sigma^0_n \) says that every \( \Sigma^0_n \) definable function maps a finite set onto a finite set. By a theorem of Kirby and Paris [9]

\[
\ldots \Rightarrow \Sigma^0_n \Rightarrow B\Sigma^0_{n+1} \Rightarrow \Sigma^0_n \Rightarrow \ldots
\]

This gives us a set of benchmarks in first-order arithmetic.

The other set of benchmarks is based on the collection of subsystems of second-order arithmetic which is used in reverse mathematics. In second-order arithmetic, the variables and quantifiers in a sentence can range over sets or relations. For example, “every nonempty subset has a least element” can be written as

\[
(\forall x)(\exists y)(x \in X \wedge y \in X \Rightarrow y \geq x).
\]

Complexity can be defined similarly. For example, the above sentence is \( \Pi_1 \). Here we only list three of the subsystems which are needed in the sequel: \( \text{RCA}_0 \) which contains \( \Sigma^0_1 \)-induction and \( \Delta^0_1 \)-comprehension: For any \( \Delta^0_1 \)-formula \( \varphi \), \( \exists x \forall n (n \in X \equiv \varphi(n)) \); \( \text{WKL}_0 \) which is \( \text{RCA}_0 \) plus weak König Lemma saying that every infinite binary tree has an infinite path; and \( \text{ACA}_0 \) which is \( \text{RCA}_0 \) plus arithmetical comprehension. Their relative strengths are known:

\[
\text{RCA}_0 < \text{WKL}_0 < \text{ACA}_0.
\]

We also need the notion of models. A model \( M \) of second-order arithmetic is a mathematical structure \((M, 0, \text{S}, +, \times, <, X)\) where \((M, 0, \text{S}, +, \times, <)\) is its first-order part and \( X \subseteq 2^M \) is the second-order part. The set variables are interpreted as members of \( X \). For example, if \( M \) is a model of \( \text{RCA}_0 \), then its second-order part \( X \) is closed under Turing reducibility and Turing join. In short, we have two measures of strength: the first-order measure which depends on the strength of induction satisfied and the second-order measure which is provided by the richness of set existence. With the concept of hierarchies available, we can recast the motivating questions as follow:

1. Suppose the colouring function \( f \) is recursive. What is the minimal syntactical complexity of a solution?
2. To which system in reverse mathematics does Ramsey’s Theorem correspond? For example, does \( \text{RT}_2^2 \) imply \( \text{ACA}_0 \)?
(3) What are the first-order consequences of Ramsey’s Theorem? For example does RT21 imply IΣ2n for n > 1?
(4) Does SRT21 imply RT22? More precisely, if a model of RCA0 whose second-order part X contains solutions for all stable colourings in X, must X contain solutions for all colourings in X?

3. Earlier Results
We now give a list of key results in the development of the subject. Some of the early studies were motivated by effective mathematics, and presented here in the language of reverse mathematics.

Theorem 3.1 (Jockusch [8]). Over RCA0,

\[\text{ACA}_0 \equiv \text{RT}_2^n \equiv \text{RT}_n^0\]

where \(n, k \geq 3\) and

\[\text{ACA}_0 \Rightarrow \text{RT}_2^n \quad \text{and} \quad \text{WKL}_0 \Rightarrow \text{RT}_2^3.\]

Theorem 3.2 (Hirst [7]). Over RCA0,

\[\text{SRT}_2^2 \Rightarrow B\Sigma_2.\]

This sets a lower bound for the first-order strength of SRT22 and hence RT22.

Theorem 3.3 (Seetapun and Slaman [13]). Over RCA0,

\[\text{RT}_2^3 \Rightarrow \text{ACA}_0.\]

Seetapun’s proof made clever use of trees, which leads to the Seetapun Conjecture:

\[\text{RT}_2^3 \Rightarrow \text{WKL}_0.\]

To determine an upper bound of first-order strength, conservation results are often used. One of the earliest conservation results was due to Harrington, who showed that WKL0 is Π11-conservative over RCA0, i.e. any Π11-statement that is provable in WKL0 is already provable in RCA0.

Theorem 3.4 (Cholak, Jockusch and Slaman [1]).

RT21 is Π11-conservative over RCA0 + IΣ2.

Corollary 3.5 (Cholak, Jockusch and Slaman [1]).

Over RCA0,

\[\text{RT}_2^2 \Rightarrow I\Sigma_3.\]

4. Recent Results
Since the work of Cholak, Jockusch and Slaman, the exact strength of RT21 became the central problem of study and a major focus of attention in reverse mathematics, with many attempts made at solving it. The extensive study created a new paradigm for the field. For example, the collection of subsystems (called the big five) widely used before the turn of the century for benchmarking the strength of a mathematical theorem was no longer sufficient. In fact, for combinatorial principles related to RT21, a linear ordering of measures to classify the strength of a system will not work. Today the picture looks rather like a “zoo”. Hirschfeldt and Shore [6] made further progress on the exact strength of many important combinatorial principles weaker than RT21. For instance, they showed that ADS is strictly weaker than RT21. However, three major questions remained open:

(1) Seetapun’s Conjecture; (2) Over RCA0, does SRT21 imply RT22? (3) Does SRT21 or RT22 imply IΣ2?

The first problem was solved by Jiayi Liu [10] who showed that

Theorem 4.1 (Jiayi Liu [10]). Over RCA0,

\[\text{RT}_2^2 \Rightarrow \text{WKL}_0.\]

However, the solution for (2) and (3) remained elusive. The most natural approach was to show that stable colourings always had a low solution (here the word “low” is a technical term in recursion theory). Or equivalently, every Σ11-set contains or is disjoint from an infinite low set. However, Downey, Hirschfeldt, Lempp and Solomon [5] showed that there is a Σ11-set D such that neither D nor \(\mathbb{N} \setminus D\) contains an infinite low subset, blocking the seemingly promising approach.

It was here that the method of nonstandard models of arithmetic came into play. We had been working on recursion theory on nonstandard models of arithmetic for more than ten years at the time. Chitat Chong [2] suggested that one should perhaps look at nonstandard models of fragments of arithmetic, because the theorem of Downey, Hirschfeldt, Lempp and Solomon relied heavily on the inductive strength of the standard model of arithmetic, with a proof involving infinite injury construction that required Σ22. On the other hand, in nonstandard models things behave
differently. For example, there is a model of \( B\Sigma_2 \) and not \( I\Sigma_2 \), first constructed by Mytilinaios and Slaman [11], in which every incomplete \( \Delta^0_2 \) set is low. In collaboration with Theodore A. Slaman, after almost ten years work this approach has turned out to be fruitful:

**Theorem 4.2 (Chong, Slaman and Yang [3]).** Over \( \text{RCA}_0 \),

\[
\text{SRT}^2_2 \Rightarrow \text{RT}^2_2 \\
\text{SRT}^2_2 \Rightarrow I\Sigma_2.
\]

**Theorem 4.3 (Chong, Slaman and Yang [4]).**

\[
\text{RT}^2_2 \Rightarrow I\Sigma_2.
\]

We end this survey with the following open questions: Does \( \text{SRT}^2_2 \) imply \( \text{RT}^2_2 \) in an \( \omega \)-model, i.e. a model with first-order part the set of natural numbers? What is the proof-theoretic ordinal of \( \text{RCA}_0 + \text{RT}^2_2 \)?

**References**

5. Rod Downey, Denis R. Hirschfeldt, Steffen Lempp and Reed Solomon, A \( \Delta^0_2 \) set with no infinite low subset in either it or its complement, *J. Symbolic Logic* 66(3) (2001) 1371–1381.