Unexpected Ramifications of Knot Theory

Mx A Gaudreau

"No arbitrariness like the choice of a metric mars the nature of a knot – a trefoil knot will be universally recognisable wherever the basic geometric conditions of our world exist. (One is tempted to propose it as an emblem of our universe.)"

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0. Introduction

A well known mathematical joke goes as follows: "a topologist is someone who doesn't see the difference between a doughnut and a coffee mug". Aside from capturing one of the main ideas of topology, the concept of genus as an invariant under diffeomorphism, it happens to capture the essence of mathematics itself; coffee.

But is this all there is to topologists?

Topology,[a] which arose in the 20th century is one of the most recent branches of mathematics. Due to its dependence on the notion of continuity, it lacks any precedent from antiquity. The fact that it exists independently of number systems, and appears to have little care for real world limitation of how surfaces can be deformed, might lead some to think that it cannot have any applications. In fact, for decades after the idea of topology was defined, under the name *analysis situs*, there hardly were any problems that mathematicians could associate with this type of analysis. This all changed with Vandermonde.

Industrialisation, cloth production, shoelaces, etc. Nobody can go a day in life without benefitting from non-trivial embedding of codimension two manifolds. That is to say: knots, tangles, and braids. Moreover, knot theory has been related to numerous research topics in fundamental sciences such as quantum physics, molecular biology and even theoretical psychology [16].

This paper is in three core sections. The first describes the inspiration and motivation for knot theory, the second concentrates on the role of knots in the development of topology, and the last talks about applications and repercussions. Throughout the text, short biographies of notable historical figures can be found. Unless otherwise stated, the source material for them is [38].

The somewhat unorthodox thesis of the following, and possible answer to the explicit question above, is the idea that the essence of mathematics is a quest for the understanding of the workings of the world.

1. Inspiration

Mathematics, as it is now understood, is a fairly recent invention. Until the 18th century shift towards axiomatic systems and logic, mathematics was mostly restricted to geometry, and focused on applications. However, knots were already an important feature of everyday life.

1.1. Functional knots

Due to the nature of strings' materials, very few knotted objects have been found by archeologists. The oldest preserved knot [16] is a fishing net, dating to 7500 BCE found near the coast of Finland. From the existence of artefacts such as sewing needles and decorative beads, it is assumed that strings and knots were invented 300,000 years ago, before fire itself. In prehistorical times, the recording of techniques to create knotted object depended on something one might call intuitive knot theory, and consisted of mnemonics one can assume were similar to the tales told to children today to teach proper shoe-lacing.

In fact, tying knots is not even an activity that is restricted to humans. Gorillas frequently use granny knots when creating nests, chimpanzees have been observed knotting and unknotting chords in captivity, and orangutans sometimes even make rope. It is thus possible that even the *homo habilis* (ancestor of the *homo sapiens*, living 2 million years ago) had rope [47].

As humans of the time had no writing systems and even less mathematical abstractions, the resulting knots had properties and technological uses which could have appeared nearly divine.

[a] The study of the properties of geometrical objects which are preserved by continuous deformations.

(A) Dürer's *First Knot*.

(B) Dürer's *Sixth Knot*.

Fig. 1. Selected examples of prints of knots by Dürer.

For example, the ancient Greeks believed that a wound bound with a "Hercules" knot would heal faster [39]. Some of these superstitions are still preserved today, through friendship bracelets, and chinese good luck knots.

1.2. Knots in art

Because of the importance of knots in technology, it is natural that they feature heavily in the art of many cultures. Moreover, the analysis of such patterns happens to benefit from a mathematical approach. This section presents some highlights of artwork representing knots, and is by no means an exhaustive survey.

Art is called modular [22] when it is composed of several basic elements (modules) that combine to create a larger piece. Modular art is easily modelled using regular graphs with a finite set of labels and some of it renders knots in one of two ways. The first way is as a connected sum, where the modules are depicting knots in an open string, connected by linking the free ends. The other option is creating a plaid where the modules are crossings or arcs. Examples of each of these are respectively celtic knots and pulli kolam, treated below. Mathematically, one can read about knot mosaics in [32].

An interesting property of planar art is its ability to encourage a three-dimensional interpretation from the viewer, even when such a figure would be impossible. This is the phenomena that Escher famously used in his art. Knot design always includes a perspective component to account for the over and under crossing information. However, one might also assume that the knot is rigid. In the case of the Borromean rings, if each component was to be a planar ring, the construction proves to be impossible, not only in its natural projection, but also in any other position of rings in space that would be equivalent to Borromean rings. However, it is possible to construct them using flat triangular rings.

Aside from these simple symbols, some art features intricate knots. For example, the First Knot of Dürer, Fig. 1, printed around 1506 was inspired by Da Vinci's work on his family seal. The figure in the print has 32 axes of symmetry, and displays an intuitive understanding of hyperbolic geometry. Dürer constructed another five knots as an exercise in variation. Aside from visual art, Dürer worked on classical geometry,

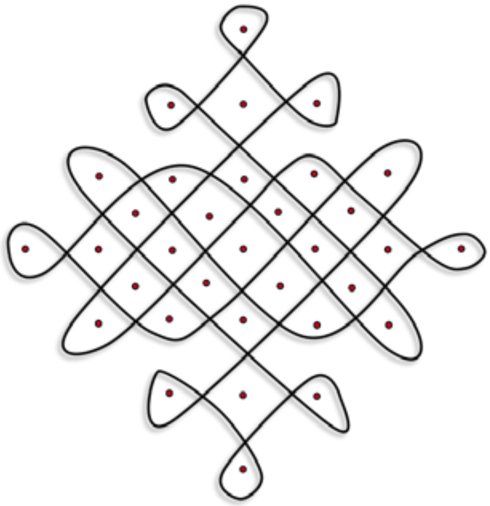


Fig. 2. Example of a Pulli Kolam from [15].

but it is unknown if he made any mathematical analysis of the knots.

Pulli Kolam is an old Tamil art form usually created on the threshold of south Indian houses. Traditionally done by women, the patterns are passed down from generation to generation and drawn from memory each morning on a cleaned ground. The first step is to place a collection of equidistant dots, then to circumscribe them in complicated orbits, often a single closed line, to create a symmetrical pattern. The patterns do not touch the dots, and have an important symbolism.

1.3. Medicine

Surprisingly, the two oldest scientific texts about knots are of a medical nature. The first one is by the Greek, Heraklas [39], who around 100 BCE, wrote a text about 18 different ways to tie slings. The work was reproduced many times until it was first illustrated in the early 16th century. The second text was produced by a French midwife named Louise Bourgeois, and mentions, in 1609, the birth of a child with a knotted umbilical cord. She believed the event to be extremely rare, and it would indeed take over two hundred years for an obstetrician to provide more details on the phenomena and claim, to much relief that it is not fatal in humans.

Since these knots are produced in an inelastic material, classical geometry can be used to understand the conditions in which a knot can be formed. Since the fetus grows, a knot that is formed early during gestation cannot be undone later. Moreover, Da Vinci gave a rule of thumb that, in humans, the cord is as long as the baby at birth. For a knot to be formed, one needs that umbilical cord to be longer than the circumference of the fetus. This proportion is often achieved, and so the estimate that about 1% of human births have knotted umbilical cord is fairly reasonable. This is all assuming that the uterus contained a single fetus. In multiple gestations, which are quite common for some species, the cords can get linked, and this situation is reputably more dangerous. More information can be found in [7].

1.4. Vortex atom theory

The Vortex Atom Model (concurrent to J J Thomson's widely accepted plum pudding model [8]) was an idea of William Thomson, also known as Lord Kelvin which was popular in the UK in 1870–1890. It was inspired by the Victorian understanding of fluid dynamics in perfect conditions and motivated by the unimaginability of the void.

A theory of ether originated in Ancient Greece, and had been supported by even Descartes and Newton, but received more notoriety after the work of Maxwell and Thomson on waves and electricity which lead to the discovery of the electron, and an early theory of particle-wave duality for the photon. And so, for wireless, particle-less energy to travel, the existence of ether was critical.

Quick facts: Lord William Thomson, Baron Kelvin of Largs (1824–1907), Irish.

Thomson was the son of a mathematician, and grew up in a strict religious household. He started attending university at the age of ten, and wrote his first paper in 1839, publishing one a short two years later. He finished the Cambridge Mathematical Tripos in 1845, and moved to Paris to work on physics, studying electrical flow under the influence of Liouville and Poisson.

Thomson took a position of Professor of Physics at the University of Glasgow, where he was a mediocre lecturer, and his initially revolutionary work turned slowly obsolete as he refused to believe his contemporaries. He was knighted in recognition for helping with the installation of a trans-atlantic telegraph cable, this accomplishment bringing him also fame, fortune and eventually the title of Baron. He served as president for the Royal Society, the British Association for the Advancement of Sciences, and the Royal Society of Edinburgh.

The goal of the vortex atom theory was none other than having an explanation for everything using only hydrodynamical equations. To quote George Fitzgerald: "If it is true, ether, matter, gold, air, wood, brains, are but different motions."

The first paper about the vortex atom by Kelvin was published in 1867. It was inspired by Tait's experiment with smoke rings designed to reproduce Helmholtz's theoretical results about the stability of the vortex tubes in an ideal fluid. In 1885, the lack of result that this theory was generating prompted the novel idea of a vortex sponge, where atoms are arrays of tightly packed vortices. William Hicks proposed further modifications, which provided explanations for gravity and electromagnetism. Kelvin's other approach was towards non-circular vortices: knots and rings. However, they could not prove that such exotic configurations were stable. Some of the good points of this theory are the understanding that such loops have a fixed mass, a vibration and a way to interact with each other. It also explained the spectral lines. Vortex theory fell out of favour because it was too flexible and didn't allow to make predictions, but only model observed phenomena.

1.5. Imagining higher dimensions

To a modern reader, imagining four dimensions is a simple exercise in formalism that vector spaces accommodate easily. Descartes introduced co-ordinate systems in the 1637 philosophy classic *Le Discours de la Méthode*, and Grassmann introduced vector algebra in 1844, giving one all the tools to imagine and manipulate a four-dimensional space [29]. However, those works were quite niche at the time, and many philosophers lacked the mathematical intuition or curiosity to understand them.

The paper "On Space of Four Dimensions" [49] was published in 1878 by Karl Friedrich Zöllner. It argued that the three-dimensional nature of space was a human construction made to account for one's observations. More so, experiences which contradict this assumption would lead to new theories for the mathematical formalism of space [5]. At the time, it was popular to connect mathematics to psychic research.

For example, if space was to be two-dimensional, a curve with a single self intersection would be *de facto* knotted and this intersection would allow no interpretation as being a crossing with one strand going over the other one, without the existence of a third dimension, in which it can be resolved. Similarly, adding a fourth dimension would allow to undo any knot without cutting it or creating self intersections, as proved by Klein. Zöllner gathered inspiration from Gauss and Kant to derive spiritual meaning from those mathematical facts. Namely, that since one can imagine the fourth dimension, it must exist.

1.6. Cryptography

Khipu[b] are a notational system of Inca origin, which were used both as record keeping and messaging system. Khipu (this word is both plural and singular) were made of strings, with a thick primary cord to which was attached many levels of secondary strings, knotted to inscribe numbers or words. The data was recorded in the type and position of the knots and strings themselves. So far, only the numerical khipu have been translated [25], the others remain a mystery. Khipu being a writing system is known because of the spanish invaders.

In the past 15 years, research in braid group cryptography has been conducted [14]. It is based on the complexity of the word problem in the braid group, and encodes messages as braids, in an entirely different way from khipu. Topological invariants could lead to new cryptography resources, or on the contrary force this line of research to close. One of the braid theoretic cryptosystems is based on the conjugacy class of the braid, which corresponds in fact to a link or knot.

2. Creation

2.1. Etymology

The study of problems that are now called topological started when Leibniz pointed out that there was a need for a different approach to geometry. One that cared less about distances and explicit formulas, but concentrated on the relative positions of objects. He dubbed this *analysis situs*, the analysis of position. However, Leibniz failed to propose problems whose solutions could be obtained by such means and it was Euler who had the privilege of stating and solving the first

[b]Sometimes also spelled quipu.

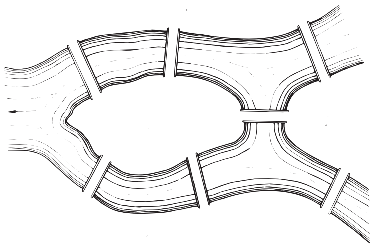


Fig. 3. Illustration of the bridges of Königsberg from [39].

topological question: Based on the map of Königsberg, Fig. 3, is there a path that goes over each of the bridges exactly once?[c]

One would have to wait until the 19th century for the new name. The word topology comes from the greek *τοπος*, meaning position, and was first used by Listing, in his paper "Vorstudien zur Topologie" in 1848. He chose to use this word instead of the latin *analysis situs* since its literal translation "geometry of position" was already in use by von Staudt, for what is now called projective geometry. Let us consider his view for the future of the field.

> Die Symmetrie des Raumes und des Bewegung bildet endlich einen ergiebigen Stoff für künftige topologische Untersuchungen, die sich theilweise schon an das bereits über die Position Vorgetragene anknpfen lassen. Wenn auch die Begriffe des Gröfse, des Masses, der geometrischen Aehnlichkeit oder Congruenz hiebei nicht aufser Acht bleibe dürfen, so treten sie doch bei der Vorstellung des räumlichen Eben-oder Gleichmasses jederzeit hinter des Begriff der modalen Raumverhältnisse zurück, wodurch die Symmetrie nitch sowohl dem Gebiete der Geometrie, als vielmehr dem der Topologie anheim fallt. Theils in der Morphologie der organisirten Wesen, theils und ganz besonders in der Krystallographie spielen die Symmetriegesetze eine wesentliche Rolle.[d]

[c]The answer is no, much to the dismay of tourists. Today, named Kaliningrad, there still is no solution to the problem using a closed path.

[d]The symmetry of space and movement finally forms a fertile material for future topological investigations which can already be partly built on and beyond the current knowledge on analysis of situation. Even though the magnitude, measure, the geometric similarity and congruence are not preserved, as they occur only during the presentation of the space and can change at any time with the concept of modal space, whereby the symmetry concerns both the areas of geometry and topology. Symmetry laws play an essential role partly in the morphology of organised beings, and especially in crystallography.

An early example in French language publications of the use of the word topology was the translation of Simony's "Ueber eine Reiheneuer Thatsachen aus dem Gebiete der Topologie" which was mentioned in the *Bulletin des sciences mathématiques at astronomiques, 2e series, tome 8, no. 2* in 1884 under the title "Sur une suite de faits nouveaux dans le domaine de la Topologie" and contained a classification of knots which can be obtained from a cutting and gluing construction.

Quick facts: Johann Benedict Listing (1808–1882), German.

As a child, he had noticeable artistic talent, which got him benefactors and income much needed for his financially unstable family. By the time he entered Gymnasium, he was more interested in mathematics than art, and thus studied architecture at the University of Göttingen as a compromise, taking classes in many sciences. There he attended lectures of Gauss, who took Listing as a student and friend, supervising his PhD thesis on differential geometry.

Listing became professor of physics a mere five years after graduating from the same institution. In 1846, Listing married Pauline Elvers, with whom he had two daughters, and struggled with debt. His research was concerned with optics, magnetism and geometry. He published very little, yet his work was often unprecedented. Unfortunately it often got popularised by other authors, for example his study of the Möbius band was written years before the work of the mathematician it is named after.

2.2. The first paper

Vandermonde's paper "Remarques sur les problèmes de situation" was named after the French translation of "*analysis situs*", and received very little recognition. It contained a modest mathematical study of knots. Here are the opening words of the paper.

> Quelles que soit les circonvolutions d'un ou de plusieurs fils dan l'espace, on peut toujours en avoir une expression par le calcul des grandeurs; mais cette expression ne seroit d'aucun usage dans le Arts. L'ouvrier qui fait une *tresse*, un

réseau, des *noeuds*, ne conoit pas par les rapports de grandeur, mais par ceux de situation: ce qu'il y voit, c'est l'ordre dans lequel sont entrelacé les fils.[e]

The initial work of Vandermonde, in 1771, was undoubtedly known to Lord Kelvin, the main investigator of the knotted atoms theory, as Maxwell had found out about it through the notebook of Gauss where the linking number integral was defined. An interesting twist is that Gauss, who had published results in *analysis situs* which were closely related to the work of Vandermonde, never cited him. There is a possibility that Gauss only read the *Histoire de L'Académie* monograph after publishing his work, or, as Lebesgue also theorised, that Gauss did not find the early work worthy of recognition since it was based on examples and application and not strict logic. Moreover, Gauss is also responsible for the first combinatorial notation of knots, which in a way, foreshadowed the duality of abstract algebra between continuous and discrete spaces.

Quick facts: Théophile Vandermonde (1735–1796), French.

Vandermonde dedicated his live to the violin until the age of 35, when the enthusiasm of des Bertins towards mathematics convinced him to give its study a try. His career in this field lasted a short three years, during which he published four papers in one of the most important journals of the time, the *Histoire de l'Académie des Sciences*. After this, he lost interest in mathematics, and devoted his time to political questions.

2.3. Enumerations

In the early years of knot theory, the most popular problem was that of enumerating all knots up to isotopy. Unfortunately, there are infinitely many knots, and infinitely many diagrams for each of them. The first part of the enumeration problem, not omitting any knot was essentially solved by Gauss already. Given that each diagram has a combinatorial expression, it suffices to list

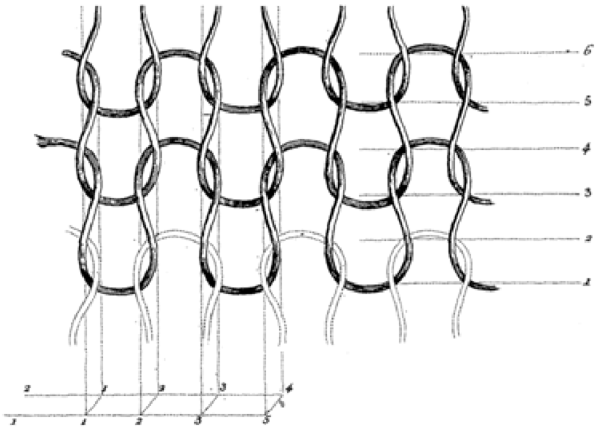


Fig. 4. Illustration from Vandermonde's paper [46].

all the possible crossing sequences.[f] However, not every possible sequence of crossing gives a knot. Various other notations were used. The work of Tait is of particular interest, since his status as a renowned scientist at the time allowed him to present his work and disseminate his ideas widely.

Quick facts: Peter Guthrie Tait (1831–1901), Scottish.

He first distinguished himself as a First Class student at Peterhouse, Cambridge. Immediately after graduating in 1852, he became a professor of mathematics at Queen's College, Belfast, then in 1858, took a position of natural philosophy (that is, physics) professor in Edinburgh. The complete published work of Tait spreads over 1000 pages, and touches topics as diverse as chemistry, thermodynamics, and quaternions, the invention of Hamilton, for whom Tait had a peculiar fascination.

Tait married Margaret Archer Porter in 1857, with whom he had four sons. He was appointed Secretary of the Royal Society of Edinburgh in 1879, and is said to have been particularly passionate in his work and personal relations, both as a loyal friend and a lifelong enemy. Tait's taste in physical experiments leaned towards the spectacular prestidigitation, making him a popular lecturer.

Tait's original goal was to list all the knots with up to ten crossings, but never reached his

[e] Regardless of the convolutions of one or many strings in space, one can always have for them an expression using calculus; but that expression would be of no use in the Arts. The worker making a *braid*, a *plaid, knots* does not think with calculus, but is concerned with position. What they see is the order in which the strings are tangled.

[f] For example, the crossing sequence of the trefoil, Fig. 5(b), is 123123, meaning that if crossings are numbered the first time they are encountered, they are crossed a second time in the same order.

goal [42] having made the false assumption that all the knots would have alternating diagrams. His technique is summarised in the three following conjectures, none of which Tait lived to see proved, even though they now have all been shown to be true.

Conjecture 1. *A reduced*[g] *alternating diagram has minimal crossing number (for the knot it represents).*

Conjecture 2. *Minimal crossings number diagrams of the same knot have the same writhe.*

Conjecture 3. *Alternating diagrams of the same knot are related by a sequence of flypes.*

Here, flype is a scottish word used to denote a type of move on minimal crossing diagrams. Once the problem of enumeration had reached 11 crossings, its exponential complexity became cumbersome and the focus shifted towards finding invariants and simplifying isotopies. The work of Haseman is therefore all the more impressive.

Quick facts: Mary Gertrude Haseman (1889–~1960), American [40].

Little is known about Haseman. As an undergraduate, she attended the University of Indiana, and spent a year of her graduate studies at John Hopkins University. Her doctoral supervisor was C A Scott at Brynn Mawr College, and her thesis, "On Knots, with a Census of the Amphicheirals with Twelve Crossings" [19] is one of her rare published works.

Here is a timeline of enumerations of knots, from [21]. Here, **a** stands for alternating, **n** for non-alternating, **c** for amphicheiral, and **o** denoted enumerations which contained omissions.

1876	7 crossings	Tait
1885	10 crossings **a**	Tait
1899	10 crossings **n**	Little
1890	11 crossings **a o**	Tait, Kirkman, Little
1914	12 crossings **c**	Haseman
1970	11 crossings **n o**	Conway
1980	11 crossings	Caudron
1983	13 crossings	Dowker, Thistlethwaite
1994	14 crossings	Hoste
1998	16 crossings	Hoste, Thistlethwaite, Weeks

2.4. Invariants

The first link invariant in the modern sense is the linking number of two non-intersecting, regular curves in $\mathbb{R}^3$, C_x, with coordinates (x_1, x_2, x_3) and C_y, with the corresponding y_i coordinates, is defined as

$$\iint_{C_x, C_y} \frac{\sum_{i=1}^{3} (y_i - x_i)(dx_{i+1}dy_{i-1} - dx_{i-1}dy_{i+1})}{4\pi(\sum_{i=1}^{3} (y_i - x_i)^2)^{3/2}}$$

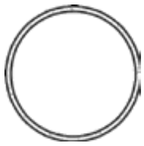
where the indices are taken modulo 3 in the range $\{1, 2, 3\}$.

It was initially defined by Gauss [16] during work on electromagnetism in 1833, but most of the work on this, and its application to knot theory was done by Boeddieker. The linking number has the ability to distinguish the Hopf link from the unlink of two components (the disjoint union of two unknots).

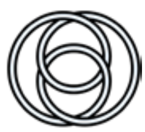
Listing[h] was the first to consider the possibility of creating an exhaustive table of knots, ordered by crossing number, and even considered braids, as inspired by his work on helices. However, he preferred the representation of knots as projections of smoothly embedded circles in space. He constructed a function on knot diagrams taking as value a polynomial in two variables. However, that function was far from an invariant as it was undefined for many diagrams, it could take different values on different diagrams of the same knot and failed to distinguish some inequivalent knot diagrams.

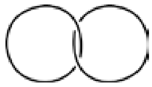
However, Listing's polynomial, and the efforts to find its shortcomings seemed to have shaped the future of knot theory as the quest for a complete and computable invariant.

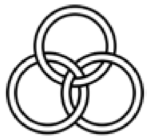
It was the work of Poincaré on fundamental groups of complexes which provided the much needed invariants. Dehn and Heegaard noted that combinatorial representations were simpler than diagrammatic ones, but they made computations of invariants, such as the Gordian (unknotting) number harder.


(A) The unknot.


(B) The trefoil knot.


(C) The figure eight knot.


(D) The Hopf link.


(E) The Borromean link.


(F) The Whitehead link.

Fig. 5. Some knots and links.

[g] That is, without any isolated crossings.

[h] Who gave his name to the unique four crossings knot in Fig. 5(c).

Quick facts: Poul Heegaard (1871–1948), Dane.

As a teenager, Heegaard was pushed towards studying mathematics, even though he had never mastered mental arithmetics, and had inherited from his late father a passion for astronomy. He obtained a master's degree from the University of Copenhagen in 1893 on algebraic curves. After which, he travelled abroad for a year, where he met Klein, who introduced him to topology. In 1896, Heegaard married Johanne Magdalene Johansen, and three years later, defended a PhD thesis which contained a counterexample to an early incarnation of Poincaré's conjecture.

Some of his main works were a book in astronomy, and a survey article, joint with Dehn, laying the ground for combinatorial topology. Until 1910, Heegaard had supported his six children with teaching at schools. Then, he became chair of mathematics at Copenhagen University, but resigned seven years later, moving on to found the Norwegian mathematical society.

In fact, it was not until Reidemeister's publication of the first book on knot theory in 1932 that knot isotopies were qualified in a workable way. Instead of relying entirely on intuition of how a string might move in space, Reidemeister proposed a set of three unoriented moves, dealing with neighbourhoods of one, two and three crossings respectively, which generate any knot isotopy. The third Reidemeister move is of special interest because it happens to be a manifestation of the Yang–Baxter equation, an important cornerstone of statistical mechanics [16]. Roughly speaking, this equation is a generalisation of the identity in the symmetric group: $(1\,2)(2\,3)(1\,2) = (2\,3)(1\,2)(2\,3) = (1\,3)$.

Quick facts: Kurt Werner Friedrich Reidemeister (1893–1971), German.

Richard Dedekin was a close friend of Reidemeister's parents, and influenced him for many years, leading him to study mathematics at university. However, his education was interrupted by World War I, where he had to serve all four years. In 1920, he took the examination to become a Gymnasium teacher, the mathematical part of which he passed with distinction. Instead of going to teach, Reidemeister became the assistant of Hecke, who would be his doctoral supervisor, for a dissertation about number theory.

Reidemeister quickly started publishing geometry work, and became associate professor at the University of Vienna in 1923, where he met and quickly married Elisabeth Wagner. They then moved to Königsberg, where Reidemeister took a chair position. There he published many books, on knot theory, geometry and algebra. However, he was forced to leave in 1933 due to his opposition to the rising Nazi party. Moving to Rome with a new, untrusting attitude, he spent the rest of his life working at various universities, publishing profusely, and keeping his political ideas private.

2.5. Definition of knot theory

As interesting and beautiful as knot tables are, they turn out to be merely a tool in this topological tale. In fact, the initial view of knots as geometric objects is less powerful than the modern definition:

Definition. *Knot theory is the study of non-trivial embeddings (that is,* knotted*) of manifolds of codimension two in $\mathbb{S}^n$ through algebraic and combinatorial methods.*

A knot is such a connected, one-dimensional manifold. A link is a union of knots which may be inseparable. The n-dimensional generalisations are called n-knots. The case we are usually concerned with [6] is $\mathbb{S}^1 \subset \mathbb{S}^3$, but long knots, $(\mathbb{R}^1 \subset \mathbb{S}^3)$ are also studied, and very recent developments include $\mathbb{S}^1 \subset \Sigma \times I$, knots in thickened surfaces.

Yet, let us return to the base case. $K \equiv \mathbb{S}^1 \subset \mathbb{S}^3$ is a *ribbon* knot if it bounds a disk with only ribbon singularities. A ribbon singularity is a self intersection in the disk such that its preimage is the disjoint union of a segment contained in the interior of the disk and one with endpoints on the boundary of the disk.

Adding a time dimension, it is possible to resolve the singularity by perturbing the disk along t. Then, K bounds a disk without singularities in $\mathbb{S}^3 \times I$. Knots with this latter property are called slice, and correspond to the intersection of a 2-manifold $\Sigma \subset \mathbb{R}^4$, with a hyperplane. "Is every slice knot ribbon?" this question is called the Slice-Ribbon conjecture and has been open ever since those notions were defined.

3. Repercussions

3.1. Fox free differential calculus

The existence of non-trivial knots, from a contemporary topological point of view was only proved in 1906 [16], by Tietze's work with fundamental

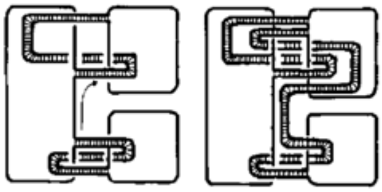


Fig. 6. Ribbon knot with singularities [37].

groups. He considered the complement of a thickened embedding (solid torus) of a knot in the three-dimensional sphere, S^3. The fundamental group of this structure is called a knot group, and it is amongst the strongest invariants developed. When paired with the peripheral structure, which describes the boundary of the manifold, it becomes a complete knot invariant.

Unfortunately, the problem of identifying equivalent group presentation is maybe even harder than that of knot classification, since the knot group and peripheral structure are but an elaborate presentation of a Gauss word for a knot. The unwieldyness of this invariant is what made Alexander's polynomial so important. Introduced in 1928, as the determinant of a matrix created from the way the arcs interacted around each crossing, it is denoted $\Delta_K(t)$, for some knot K, and variable t.

Many of the computations one can do for a knot are based on the idea of a *skein relation*. Considering the knot to be a way of connecting crossings together lead to asking about the relations between knots which differed only at a single crossing. Given four strands in a knot diagram L, there exists four ways[i] they could be connected, denoted L_+, L_-, L_0 and L_∞, as per Fig. 7.

The connected sum, which was mentioned in the section on modular art, demonstrates some of the good behaviour of Alexander's eponymous polynomial. In fact, it is multiplicative under this operation. Unfortunately, this is not a complete invariant, as it takes value 1 on some knots different from the unknot,[j] nor does it distinguish between mirror images. It was the first invariant to be presented in the form of a skein relation, that is a linear relation between the polynomials of knots which were identical except for one crossing. Although that relation was present in the original paper, the matrix approach remained the most popular until Conway's rediscovery.

[i]In fact, there exists a fifth one, arising from a field known as virtual knot theory, which is unfortunately beyond the reach of this work.

[j]The Alexander polynomial of the unknot has to be one by the multiplicative property, since any knot is the connected sum of itself with the unknot.

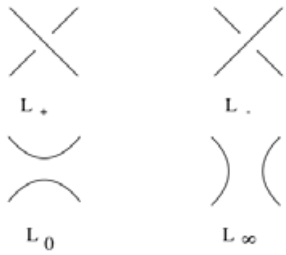


Fig. 7. Crossings and resolutions assuming the strands are oriented towards the right in the first two cases [39].

For over 20 years, the true nature of Alexander polynomial remained conjectural at best. Eventually, a third definition for the polynomial emerged, where the matrix arose as the derivatives, in a certain sense, of the relations in a presentation of the knot group. The importance of free differential calculus is best explained the words of its creator, Fox [11] himself.

> As the calculus developed, it became increasingly clear that *free differentiation is the fundamental tool for the study of groups defined by generators and relations*. It is closely connected with several of the significant modern developments of algebra and topology and, in fact, reveals hitherto unobserved relations between them.

After the algebraic revolution, which gave invariants, came a computer revolution which yielded computability. In 1970, Conway revisited and completed knot and link tables, by computing many invariants with the aid of computer programs.

Quick facts: John Horton Conway (1937–), British.

At the age of eleven, Conway was already stating that his goal in life was to become a mathematician. He obtained a B.A. from Cambridge in 1959, and a PhD from the same institution after five years of research in number theory. Immediately after graduating, he took a position as a lecturer.

Conway's mathematical contributions tend to be regarded as bizarre, as his interest in group theory and the game of Go merged towards the invention of surreal numbers, the Game of Life, and Monster groups. Since 1986, he has been working at Princeton University.

3.2. A coincidental polynomial

The P-polynomial was introduced in 1985 in a note, [12] by Hoste, Ocneanu, Millett, Freyd, Lichorish, and Yetter, whose initials form the acronym, HOMFLY, often used to name the two-variable polynomial. The large number of coauthors on that paper was the result of a surprising coincidence. Originally, four separate papers were submitted to the American Mathematical Society, within a very short time frame, and describing the very same polynomial. It was the editors' suggestion that all six authors collaborate together to publish the result.

Simultaneously, Przytycki and Taczyk developed the same invariant [16]. Having submitted their independent work only two years later, their contribution is only sometimes acknowledged by the use of the *HOMFLY-PT* acronym.

This polynomial is notable for its ability to specialise, by changes of variables to both the Alexander, and the Jones polynomial.[k] Similarly to the Jones polynomial, and its unnormalized[l] relative, the Kauffman bracket, the P-polynomial can be computed from a state model formula based on the L_0 and L_∞ smoothings.

3.3. Statistical mechanics

Statistical mechanics is the study of equilibrium in mechanical systems. It often focuses on the second law of thermodynamics, that entropy in a closed system is constant, but lacks a unifying theoretical base. Instead, statistical mechanics is trying to prove things such as why the Poincaré lemma,[m] are not observable. To do so, one assigns to mutually exclusive states of a system a probability of being observed, and considers the expected behaviour over time [13].

Jones was awarded a Fields' Medal in 1990, for relating the previously separate fields of knot and braid theories, through their respective interactions with statistical mechanics. The 1986 article by Jones introduced his polynomial valued knot invariant, which detects handedness in some knots. The quest for a non-trivial knot with trivial Jones polynomial is open to this day, suggesting that the polynomial detect the unknot.

The Jones polynomial was instrumental in the proof of Tait's first and second conjectures in 1987, independently by Kauffman [27], Murasugi [35, 36], and Thistlethwaite [43, 44].

This invariant inspired two entirely distinct generalisations. The first one is the one discussed in the previous subsection. The second one is called the *F*-polynomial, and was discovered by Kauffman. Unlike the skein relations for the Alexander polynomial, Kauffman's work is based on a state model which express each knot diagram as a sum of the diagrams that can be obtained from smoothing each crossing as either L_0 or L_-. The *F*-polynomial is a two-variable version of the bracket polynomial. Finally, Tait's third conjecture was proved using that polynomial invariant by Menasco and Thistlethwaite in 1991 [33].

Kauffman is a major player in building the relations between knot theorists and physicists. In 1991, he published the book *Knots and Physics*, which paved the way for the *Series on Knots and Everything*, of which many books are referenced herein.

3.4. Braid theory

A braid is an element in the group consisting of finite loops in a punctured disk, up to homotopy. There is thus one braid group for each natural number n, called B_n and corresponding to the fundamental group of the disk with n punctures. The unital difference comes from the original interpretation of the braids as elements of the fundamental group of the space of non-degenerate polygons in $\mathbb{R}^2$. However, clockwise loops around each puncture can be identified with a generator σ_i, while counterclockwise ones are denoted σ_i^{-1}, and representing words as stacks of the generating tangles gives the braided visual.

The relations that hold for this group are quite natural and usually called invertability ($\sigma_i\sigma_i^{-1} = 1$), far commutativity ($\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $|i - j| \geq 2$), and the appropriate form of the Yang–Baxter equation ($\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$).

[k] The Jones polynomial is briefly discussed in the next section. To see how it is defined, see Jone's recent survey [23], which takes multiple approaches towards the invariant.

[l] In fact, the Kauffman bracket can be considered to be an invariant of the equivalence class of knot diagrams where all one of the Reidemeister moves are allowed. These classes are called framed knots, and model in an effective way the behaviour of a knotted belt, where a Möbius band would not be considered equivalent to an annulus.

[m] Which would predict that indeed, the original cloud of tea should eventually reform in a teacup.

Quick facts: Emil Artin (1898–1962), Austrian. Until the age of sixteen, Artin was mostly interested in music and chemistry, and had no particular talent for mathematics. His university studies were briefly interrupted by WWI, but in 1919, he started at the University of Leipzig in mathematics. In a mere two years, he obtained a doctorate, under the supervision of Herglotz. He then became Privatdozent at the University of Hamburg, where he proved a passionate lecturer of mathematics and physics.

Artin was a particularly active researcher, solved one of Hilbert's problems, made wild but accurate conjectures, developed the theory of braids and greatly advanced many other abstract algebra topics. After marrying Jewish Natalie Jasny in 1929, his focus slowly turned towards his family. Jasny's religion forced the family to move to the USA in 1937, where Artin devoted his life to supervising students and writing books.

There is a natural map from braids to links, obtained by taking their closures, that is, identifying the top first strand from the left with the bottom first strand on the left, and so on. The kernel of this map is described by the Markov theorem[n] stating that two braids close to the same knot if and only if they are related by a chain of conjugation and stabilisation moves.

As braids and an intrinsically algebraic object and closure is a map onto the set of knots, this approach contributed to making algebraic methods more popular than combinatorial ones.

In one of Artin's original papers, [2], establishing braid theory, he paints a vivid portrait of the complexity of hand calculations which has to be shared.

> Although it has been proved that every braid can be deformed into a similar normal form the writer is convinced that any attempt to carry this out on a living person would only lead to violent protests and discrimination against mathematics. He would therefore discourage such an experiment.

He also coined the term "enormously simple" and was not one to fall in the common trope of calling something trivial lightly.

[n]Of A A Markov Jr, son of the mathematician who gave his name to the matrix chains.

Quick facts: Joan Sylvia Lyttle Birman (1927–), American.
In primary school, Birman was fascinated with patterns, and became interested with geometry in high school. Her family pushed her to pursue academia, hence she took mathematics in college, graduating with a BA in 1948. Until 1961, she worked in industry, and founded a family with physicist Joseph L Birman. Then, she took up graduate studies at the Courant Institute, finding an interest in pure mathematics.

Her doctoral supervisor was Magnus, and the work that she started as a thesis, on braid groups and the mapping class groups was only the first stone in a text that would become an instant classic. Birman worked at the Stevens Institute, Princeton and Barnard College, retiring in 2007, after an award winning career, which even included creating a prize in honour of her late sister.

3.5. Knots for everything

As knot theory started as an attempt to find a theory of everything, a simple scheme that would explain the nature of the universe itself, it so happened that, as false as the first idea was, the current theory was greatly inspired by the independent development of knot theory over two centuries.

Matter is made of atoms, which themselves are divided in electrons,[o] protons and neutrons.[p] The *Standard Model* is the description of the 12 elementary particles: quarks (up, down, strange, charm, top, bottom), electron, muon, tauon, and their partnered neutrinos, the last six collectively known as leptons. And their interactions through the four fundamental forces: gravity, electromagnetism, weak and strong nuclear forces. The forces are realised by an exchange of particles, respectively the graviton, photon, W or Z boson and the gluon.[q]

The theory of gravity at a small scale is not well explained by the standard model. String theory proposes that all the fundamental particles are made of "strings". Even within the Standard Model (SM), more than four dimensions of

[o]A type of lepton, hence elementary.
[p]Both composed of quarks.
[q]Collectively known as bosons.

spacetime are technically needed: particles not only have a position and a velocity, but also a mass, electric charge and spin.

The mathematical background of the SM is quantum field theory, and this allows consistency between quantum mechanics and the special theory of relativity. However, general relativity cannot be derived within this framework. Strings can be closed or open, and their vibration modes are seen as different masses and spins, hence describe the elementary particles of the standard model, fundamentally including the graviton, giving a theory of quantum gravitation.

The first string theory was called Bosonic String Theory, because it didn't include the fermions. These can be accounted for by introducing supersymmetry, which yields from BST three consistent superstring theories. Combining them together, two heterotic string theories have been formulated. Note that when string theory is described to be 10-dimensional, six of these dimensions are in fact compact.

It is only in 1995 that American physicist Edward Witten proposed a solution to this branching: seeing strings as slices of a larger manifold. He called this M-theory. The visual, of slicing a two-dimensional manifold with a hyperplane to obtain a one-dimensional object should remind you of the slice knot construction. This could have been entirely coincidental, had Witten not been a leading knot theorist.

Quick facts: Edward Witten (1951–), American.

Witten's first interest was in history and linguistics, in which he obtained a bachelor thesis from Brandeis in 1971. Without doubt encouraged by his theoretical physicist parent, Witten then attended Princeton for a Master's degree in applied mathematics, and a PhD in physics under the supervision of D Gross.

Witten has held positions at many renowned American university, while working on reuniting theoretical physics with geometry and abstract algebra. In 1990, he became the first physicist to be awarded a Fields Medal, a crowning achievement amongst an impressive number of prizes and recognitions. He is currently married to fellow physicist and Princeton professor Chiara Nappi.

Dijkgraaf summarises well the relation between many of the notions that have been presented in the following table, where α' denote the *stringyness* and λ the quantum correlation.

	$\lambda = 0$	$\lambda > 0$
$\alpha' > 0$	conformal field theory, strings, quantum cohomology	M-theory, string fields, branes, non-commutative geometry
$\alpha' = 0$	quantum mechanics, particles, combinatorial knot invariants	quantum field theory, fields, low-dimensional manifolds

3.6. Molecular biology

DNA is a long molecule, formed in an oriented, flexible double helix which when considered as one-dimensional, takes three configurations: catenated, super coiled or knotted. These configurations are obtained when various enzymes act on the protein. Topology has been applied to the study of DNA knotting and recombining. Experiments using gel electrophoresis can calculate the linking number and writhe of the knots formed by DNA. Calculating the action of enzymes on those topological invariants allows one to understand the chemical pathway better. However, the reason for which molecules for knots is unknown. The current hypothesis is that this increases the molecule's stability. The reference [34] is a short survey article about the applications of knot theory in the study of DNA, written in a style accessible to mathematicians.

The history of the discovery of the structure of DNA is a long and winding tale. Like every protein, it folds and coils on itself, obscuring the base helix structure. Some details of early DNA research can be found on the following biography, of one of its sometimes forgotten actors.

Quick facts: Rosalind Franklin (1920–1958), British [3].

Franklin excelled in sciences in school, and was strong headed enough to oppose her family's wishes, and dedicate her life to science at the age of 15. She graduated from Cambridge in 1938, and while working in industry, developed gas masks, published papers, and earned a PhD in Physical Chemistry from Cambridge.

In Paris, she worked with J Mering on X-ray chromatography, a science that she would bring back to England and use to discover the double helix configuration of DNA. However, her work was impeded by sexism, and stolen by a fellow researcher who would carry on to win a Nobel Prize. Her later work instituted the field of virology.

3.7. Topology of the mind

Psychoanalysis is a theory and therapy developed by Sigmund Freud. He believed that the problems of the conscious mind were caused by the unconscious and that exposing them, would lead to solutions.

For Lacan, the mind of humans have three components: the Real, the Symbolic and the Imaginary. He explained the dynamics of desire by modelling each part as a solid torus, since they each have a single void, and arranged them to form Borromean rings, as they each superpose the other two: $R \supset S \supset I \supset I \supset$ ....

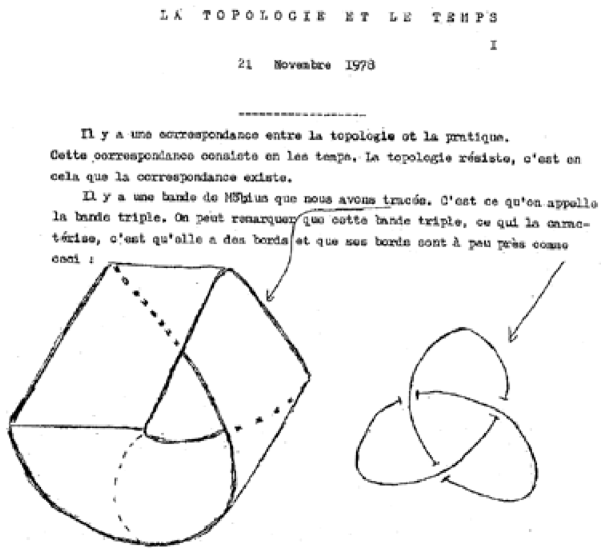
Quick facts: Jacques-Marie Émile Lacan, (1901–1981), French [4].

Raised in a catholic family, Lacan left the church as a teenager, more interested in the works of Leibniz and Spinoza. During his time at medical school, he frequents both surrealists[a] and members of the *action françcaise*.[b] In 1932, he obtained a doctorate in psychiatry and an MD.

Lacan's views would be strongly contested at the time, and, having created a chasm in the french psychology society, he turned to topology in the 70s, eventually defining himself a linguist, and died of a cancer he refused to cure.

[a]Artistic and political movement advocating for the importance of dreams, the subconscious, and automatism.
[b]A nationalist, right-winged organisation.

Some of his other work concerned the three-twist Möbius band, which he correctly states to be spanning the trefoil (as a surface bounded by it without singularities, not as a ribbon, which can only be a disk), and a generalisation of the Borromean rings to five circles, that would allow him to further his theory of the topology of the mind. However, his requirement for this link would be that removing any two circles should leave the other three unlinked. However, no such link exists, as the Brunnian property is that removing any one ring frees the others (realisable with any number of component), and the Borromean property is such that any two rings are unlinked (realisable only with an odd number).


LA TOPOLOGIE ET LE TEMPS

I

21 Novembre 1978

Il y a une correspondance entre la topologie et la pratique. Cette correspondance consiste en les temps. La topologie résiste, c'est en cela que la correspondance existe.

Il y a une bande de Möbius que nous avons tracée. C'est ce qu'on appelle la bande triple. On peut remarquer que cette bande triple, ce qui la caractérise, c'est qu'elle a des bords et que ses bords sont à peu près comme ceci :

Fig. 8. Notes from Lacan's last seminar.

Philosophical or psychological questions have a history of motivating mathematical research that date back to Greek antiquity, but since the separation of foundations of mathematics as a topic in its own right, the relation has been abandoned. This link that Lacan was looking for might be one of the last examples of such a question.

4. Conclusion

In this text, the importance of knots in the technology, science, and arts has been presented. Simultaneously, the development of pure mathematics has allowed people to formalise knot theory, and has contributed to all the fields where knots are important.

Through the meanderings of the development of knot theory as a stand alone field of mathematics, many techniques were discovered which contributed to abstract algebra, mathematical physics, cryptography, and algebraic topology. The classification of knots in itself is an important catalogue of subtly different manifolds.

Let us here return to this essay's main question. What is the motivation for the study of knot theory? It is impossible to question every knot theorist on the planet, and one can only consider the broad stokes of the research fields. Some of the applications are used to answer the oldest questions of humanity. Where does the world come from? What is matter made of? How did Life come to be? Those are all motifs found in foundational texts of cultures around the world. Respectively, statistical physics, M-theory, and molecular biology each provide a hypothesis which could answer those questions.

So, if knot theory itself is no longer a quest for the Ultimate Answer [1], it remains an inspiration for people seeking it, and in some people's opinion, is a part of the Answer itself.

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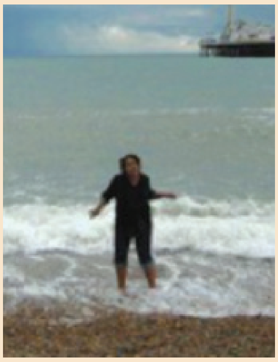

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