Problem Corner 6.1

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Welcome to Problem Corner 6.1. If you solve any of these problems, please send your solutions to APMN@wspc.com by 30 September 2016. A *book prize* will be awarded to the person with the best submission. The solutions will be posted in a future Problem Corner.

Problem 1 — Right Angles

Let k be the number of interior right angles in an n-sided polygon. Note that a $270°$ interior angle does not count as an interior right angle.

Prove that

$$k \leq \frac{2}{3}(n+2).$$

Problem 2 — Inequality

Prove that if x, y, z are distinct positive real numbers, then

$$\frac{x(y-z)}{y+z} + \frac{y(z-x)}{z+x} + \frac{z(x-y)}{x+y} \neq 0.$$

Problem 3 — Random Numbers

A computer is generating random numbers uniformly chosen from the interval $[0, 1]$. This process continues until the sum of all numbers generated is at least 1.

On average, how many numbers are required to reach a sum of at least 1?

Problem Corner 5.2 Solutions

Problem 1 — Square Digits

For any integer $n \geq 1$, define $f(n)$ as the number $11\cdots144\cdots4$ where the digit 1 occurs n times and the digit 4 occurs $2n$ times.

Find all values of n for which $f(n)$ is a perfect square.

Solution: We claim the only perfect square is $f(1) = 144 = 12^2$. First note that, for $n \geq 2$, $f(n)$ ends with at least four 4's. Since 10^4 and 4448 are both multiples of 16, we can deduce that for $n \geq 2$,

$$f(n) \equiv 12 \mod 16.$$

It suffices to check that no perfect square can be $12 \mod 16$.

Every even square must be of the form $(4k)^2$ or $(4k+2)^2$. Checking the two cases,

$$(4k)^2 = 16k^2, \quad (4k+2)^2 = 16k^2 + 16k + 4,$$

we see that neither of them are $12 \mod 16$. Thus $f(n)$ is never a perfect square for $n \geq 2$.

Problem 2 — Rows and Columns

Some number of pebbles are placed on a 7×7 chessboard, such that each pebble is placed inside a unit square and each unit square has at most one pebble.

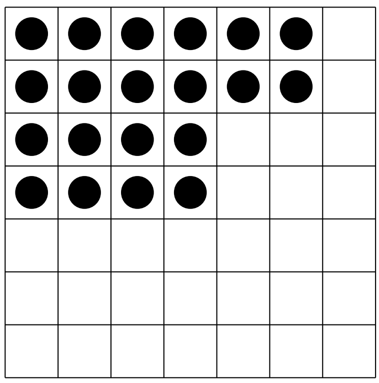
If each row and column of the chessboard contains an even number of pebbles, how many pebbles can there be altogether?

Solution: The number of possible pebbles are given by the following criteria:

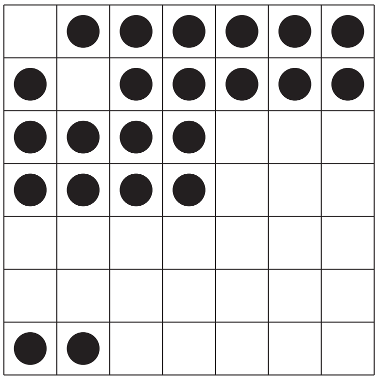
$$n \text{ even}, \quad 0 \leq n \leq 42, \quad n \neq 2.$$

It is clear that n must be even. If $n = 2$, the two pebbles cannot simultaneously be in the same column and the same row. Hence there will be a row or a column with only one pebble, so $n = 2$ is not possible. Also, since every row must be missing at least one pebble, we cannot have more than $7^2 - 7 = 42$ pebbles.

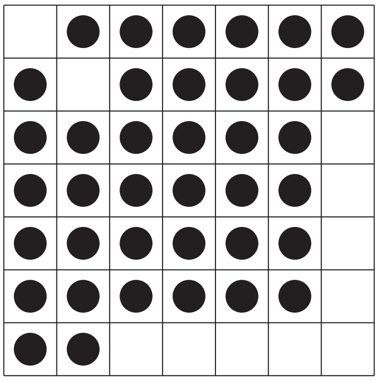
For $n = 0, 4, 8, \ldots, 36$, we can construct an example by successively placing 2×2 blocks of pebbles, starting from the top left corner. The case for $n = 20$ is shown below.



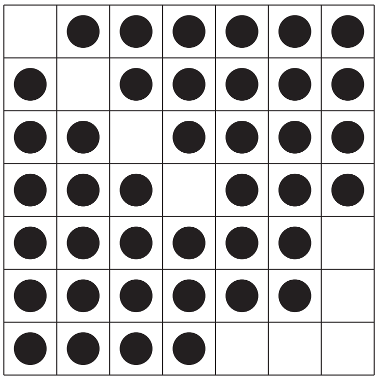
For $n = 6, 8, \ldots, 38$, begin with the construction for $n - 2$. Now remove two pebbles near the top left corner ($(1, 1)$ and $(2, 2)$) and place four pebbles on the bottom and right edges ($(7, 1), (7, 2), (1, 7)$ and $(2, 7)$). This is demonstrated for $n = 22$ in the following diagram.

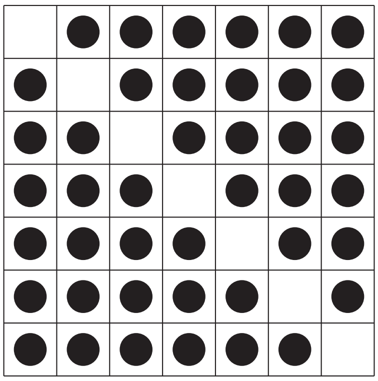


This construction can be used for up to $n = 38$.



Finally, the constructions for $n = 40$ and 42 are shown below.



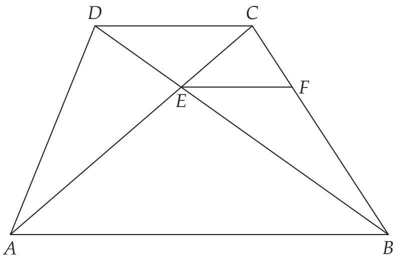


This concludes the solution as we have checked all the cases.

Problem 3 — Parallel Length

Let $ABCD$ be a trapezium such that AB is parallel to CD. Let E be the intersection of the diagonals. Let F be a point on BC such that EF is parallel to both AB and CD.

Find the length of EF in terms of the lengths AB and CD.



Solution: Since $AB \| CD \| EF$, It is easy to verify the following pairs of similar triangles,

$$\triangle BCD \sim \triangle BFE, \quad \triangle CAB \sim \triangle CEF.$$

They imply the following length ratios,

$$\frac{BF}{BC} = \frac{EF}{DC}, \quad \frac{CF}{CB} = \frac{EF}{AB}.$$

Adding the ratios yields

$$\frac{EF}{DC} + \frac{EF}{AB} = \frac{BF}{BC} + \frac{CF}{CB} = 1.$$

Finally, rearranging the previous equation gives the desired result,

$$EF = \frac{AB \cdot CD}{AB + CD} = \frac{1}{\frac{1}{AB} + \frac{1}{CD}}.$$

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